

Perfect Orderings on Bratteli Diagrams

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Abstract

Given a Bratteli diagram B , we study the set \mathcal{O}_B of all possible orderings ω on a Bratteli diagram B and its subset \mathcal{P}_B consisting of *perfect* orderings that produce Bratteli-Vershik dynamical systems (Vershik maps). We give necessary and sufficient conditions for ω to be perfect. On the other hand, a wide class of non-simple Bratteli diagrams that do not admit Vershik maps is explicitly described. In the case of finite rank Bratteli diagrams, we show that the existence of perfect orderings with a prescribed number of extreme paths affects significantly the values of the entries of the incidence matrices and the structure of the diagram B . Endowing the set \mathcal{O}_B with product measure, we prove that there is some j such that almost all orderings on B have j maximal and minimal paths, and that if j is strictly greater than the number of minimal components that B has, then almost all orderings are imperfect.

1 Introduction

Bratteli diagrams, which originally appeared in the theory of C^* -algebras, have turned out to be a very powerful and productive tool for the study of dynamical systems in the measurable, Borel, and Cantor setting. During the last two decades, diverse aspects of Bratteli diagrams, and dynamical systems defined on their path spaces, have been extensively studied, such as measures invariant under the tail equivalence relation, measurable and continuous eigenvalues, entropy and

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orbit equivalence of these systems. We refer only to a recent survey by Durand [D10] where the reader will find more references on this subject.

The importance of Bratteli diagrams in dynamics is based on the remarkable results obtained in the pioneering works by Vershik, Herman, Giordano, Putnam, and Skau [V81], [HPS92], [GPS95]. Among other fundamental results, it was proved that any minimal homeomorphism of a Cantor set can be realized as a homeomorphism (usually called a *Vershik* or *adic* map) defined on the path space of a simple Bratteli diagram. The key tool in this type of results is the concept of using a ‘proper’ order on a Bratteli diagram to represent the dynamics. Later on, Medynets proved that this result holds for aperiodic homeomorphisms of a Cantor set [Me06] where the corresponding diagrams are non-simple - thus many of these notions and results make sense for non-simple diagrams. As soon as a diagram has strictly more than one minimal component, non-proper orderings have to be considered.

To each Bratteli diagram B we associate a set \mathcal{O}_B of all possible orderings on B . It is worth commenting here that we use in this paper the term ‘ordering’ (instead of more usual ‘order’) to stress the difference between the case of ordered Bratteli diagrams, when an order is rigorously attached to the diagram, and Bratteli diagrams with variable orderings - the latter is exactly our case. An ordering $\omega \in \mathcal{O}_B$ is completely defined on B if for each vertex v of B one enumerates $r^{-1}(v)$, the set of edges e whose range is v . We call an ordering ω *perfect* if it defines a Vershik map φ_ω and denote by \mathcal{P}_B the set of all perfect orderings. Our original motivation for this work was the following *problem*: given a Bratteli diagram B and an ordering ω on B , determine under what conditions on (B, ω) the Vershik map φ_ω exists. Do there exist simple criteria that would allow us to distinguish perfect and non-perfect orderings? We give affirmative answers to this question in this paper, in Proposition 3.2.

For a simple Bratteli diagram, the set of proper orderings, a non-empty subset of \mathcal{P}_B , corresponds to a large collection of strongly orbit equivalent Vershik maps. On the other hand, if B is a simple rank d diagram, with $d > 1$ then $\mathcal{P}_B \neq \mathcal{O}_B$. The case of non-simple Bratteli diagrams is more complicated. An example of a non-simple diagram B such that $\mathcal{P}_B = \emptyset$ was first found by Medynets in [Me06]; in the present work, we clarify the essence of Medynets’ example and describe a wide class of non-simple Bratteli diagrams with no perfect ordering in Propositions 3.25 and 3.26.

It is not hard to see that the set \mathcal{O}_B can be represented as a product space and the product topology turns it into a Cantor set. Moreover, since it is natural to assume that orders on $r^{-1}(v)$ have equal probability, we can consider a uniformly distributed product measure μ on \mathcal{O}_B . In this context, the following questions are interesting to us. Given a Bratteli diagram B , what can be said about the set \mathcal{O}_B

and its subset \mathcal{P}_B from topological and measurable points of view? Is the map $\omega \mapsto \varphi_\omega$ continuous?

Recall that for finite rank Bratteli diagrams the number of vertices at each level is bounded. In this article we restrict ourselves mostly to finite rank diagrams, and indeed some of our results do not clearly generalize to infinite rank diagrams. At first glance it seems surprising, but the fact that there is a perfect ordering ω on a finite rank diagram B says a lot about the structure of the diagram - in particular, about the entries of the incidence matrices F_n . Moreover, if additionally we know that the perfect ordering has several maximal and minimal paths, then even more information about the structure of B can be deduced. This is explicitly described in Theorem 3.19. The simplest case is if a perfect ordering ω has d maximal and d minimal paths and is supported on a simple finite rank d Bratteli diagram B . Then ω determines almost completely B 's incidence matrices (F_n) (Theorem 3.15). A consequence of Theorem 3.19 and Remark 3.20, along with the fact that aperiodic Cantor homeomorphisms can be represented as adic systems, is that non-minimal aperiodic dynamical systems do not exist in abundance.

For finite rank diagrams, extremal (i.e. maximal and minimal) paths can be made ‘vertical’ by telescoping. This allows us to define new notions related to a finite rank diagram. They are the *language*, *skeleton* and *associated graphs*. We remark that these notions can be generalised to infinite rank diagrams; however the fact that we cannot necessarily telescope paths to make them vertical means that the corresponding definitions are more technical, especially notationally. When an ordering ω is chosen on a diagram B which has d vertices at each level, then we can consider the set of all words over a d -letter alphabet formed by sources of finite paths with the same range. This set of words defines the language of the ordered diagram (B, ω) . We use the language to characterize whether or not ω is perfect. If an ordering ω is given on a finite rank diagram B , then we can consider the subset of edges formed by maximal and minimal edges. This is the skeleton defined by ω (although the notion of a skeleton only requires partial information about an ordering). We use this notion as well as the notion of associated graphs to characterize finite rank diagrams that have a prescribed set of extreme paths.

The organization of the article and main results are as follows. Section 2 contains the definitions of Bratteli diagrams, orderings, Vershik maps, etc., which are given in the most general form for non-simple diagrams. Although these definitions are well known, we include this material to fix our notation and make the paper self-contained. We also show that the set of perfect orderings \mathcal{P}_B and its complement are dense in \mathcal{O}_B (Proposition 2.14) and have empty interiors provided that there are at least two vertices at each level. The relation $\mathcal{O}_B = \mathcal{P}_B$ holds only for diagrams with one vertex at infinitely many levels (Proposition 2.11).

Section 3 is devoted to the study of orderings on finite rank diagrams. First of all we define the notions of language, skeleton, and associated graphs. They are used to study the structure of diagrams and perfect orderings. We formulate some criteria for an ordering to be perfect (Propositions 3.2 and 3.4, and Lemma 3.3). We give sufficient conditions for a Bratteli-Vershik system to be isomorphic to an odometer (Propositions 3.5 and 3.23). We also find necessary and sufficient conditions on entries of incidence matrices, that guarantee the existence of a perfect ordering with prescribed sets of maximal and minimal paths (Theorem 3.19). Based on our analysis of the language of perfect orderings, we find a class of diagrams that do not admit any perfect orderings (Propositions 3.25 and 3.26). Lemma 3.3 tells us that the telescoped image (B', w') of (B, ω) is perfect if and only if (B, ω) is perfect. If B has no perfect orderings, are there any perfect orderings on B' ? Presumably, the answer should be negative, but we could not prove it.

In Section 4, we endow the set \mathcal{O}_B with the uniform product measure, and study questions about the measure of specific subsets of \mathcal{O}_B . We show, in Theorem 4.1 and Corollary 4.3, that for a finite rank d diagram there is some $1 \leq j \leq d$ such that almost all orderings have exactly j maximal and j minimal paths. Whether for diagrams with isomorphic dimension groups the j is the same is an open question. We give necessary and sufficient conditions, in terms of the incidence matrices of B , for verifying the value of j , and show that $j = 1$ for a large class of diagrams which include linearly recurrent diagrams. We show in Proposition 4.5 that if B is simple and $j > 1$, then a random ordering is not perfect.

2 Bratteli diagrams and Vershik maps

2.1 Main definitions on Bratteli diagrams

In this section, we collect the notation and basic definitions that are used throughout the paper; these definitions are standard, but we include them to fix our notation. More information about Bratteli diagrams can be found in the papers [HPS92], [GPS95], [DHS99], [Me06], [BKM09], [BKMS10], [D10] and references therein.

Definition 2.1. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i \geq 0} V_i$ and the edge set $E = \bigcup_{i \geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i such that

- (i) $V_0 = \{v_0\}$ is a single point;
- (ii) V_i and E_i are finite sets;
- (iii) there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

The pair (V_i, E_i) or just V_i is called the i -th level of the diagram B . A finite

or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively. For $m < n$, $v \in V_m$ and $w \in V_n$, let $E(v, w)$ denote the set of all paths $\bar{e} = (e_1, \dots, e_p)$ with $s(e_1) = v$ and $r(e_p) = w$. For a Bratteli diagram B , let X_B be the set of infinite paths starting at the top vertex v_0 . We endow X_B with the topology generated by cylinder sets $U(e_j, \dots, e_n) := \{x \in X_B : x_i = e_i, i = j, \dots, n\}$, where $(e_j, \dots, e_n) \in E(v, w)$, $v = s(x_1) \in V_{j-1}$ and $w = r(e_n) \in V_n$. Then X_B is a 0-dimensional compact metric space with respect to this topology. We will consider such diagrams B for which the path space X_B has no isolated points. This means that for every $(x_1, x_2, \dots) \in X_B$ and every $n \geq 1$ there exists $m > n$ such that $|s^{-1}(r(x_m))| > 1$. Here and thereafter $|A|$ denotes the cardinality of the set A .

We will assume that the following *convention* always holds: our diagrams are not disjoint unions of their subdiagrams. Here $B = (V, E)$ is a *disjoint union* of $B^1 = (V^1, E^1)$ and $B^2 = (V^2, E^2)$ if $V = V^1 \cup V^2$, $V^1 \cap V^2 = \{v_0\}$ and $E = E^1 \sqcup E^2$.

Given a Bratteli diagram $B = (V, E)$, the incidence matrix $F_n = (f_{v,w}^{(n)})$, $n \geq 1$, is a $|V_{n+1}| \times |V_n|$ matrix whose entries $f_{v,w}^{(n)}$ are equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, i.e.,

$$f_{v,w}^{(n)} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

For convenience, we will assume in some cases that $F_0 = (1, \dots, 1)^T$ though this assumption is not restrictive for our purposes.

Observe that every vertex $v \in V$ is connected to v_0 by a finite path and the set $E(v_0, v)$ of all such paths is finite. Set $h_v^{(n)} = |E(v_0, v)|$ where $v \in V_n$. Then

$$h_v^{(n+1)} = \sum_{w \in V_n} f_{v,w}^{(n)} h_w^{(n)} \text{ or } h^{(n+1)} = F_n h^{(n)}$$

where $h^{(n)} = (h_w^{(n)})_{w \in V_n}$.

A Bratteli diagram $B = (V, E)$ is called *simple* if for any level n there is $m > n$ such that $E(v, w) \neq \emptyset$ for all $v \in V_n$ and $w \in V_m$. If $F_n = F_1$ for all $n \geq 2$, then the diagram B is called *stationary*. Hence, the incidence matrix F_1 carries complete information about a stationary Bratteli diagram. If $|V_n| \leq d$ for all $n \geq 1$, then B is called of *finite rank*. Let d be the smallest integer such that there are infinitely many levels in B with d vertices: we then say B has *rank* d . If a minimal Cantor dynamical system (Y, T) admits a representation by a Bratteli diagram $B = (V, E)$ with $|V_n| = 1$ for all levels n , then T is called an *odometer*.

Definition 2.2. For a Bratteli diagram B , the *tail (cofinal) equivalence* relation \mathcal{E} on the path space X_B is defined as $x \mathcal{E} y$ if $x_n = y_n$ for all n sufficiently large where $x = (x_n)$, $y = (y_n)$.

Let $X_{per} = \{x \in X_B : |[x]_{\mathcal{E}}| < \infty\}$. By definition, we have that $X_{per} = \{x \in X_B : \exists n > 0 (|r^{-1}(r(x_i))| = 1 \ \forall i \geq n)\}$. A Bratteli diagram $B = (V, E)$ is called *aperiodic* if $X_{per} = \emptyset$, i.e., every \mathcal{E} -orbit is countably infinite.

In the paper, we constantly use the *telescoping* procedure for a Bratteli diagram. Roughly speaking, in order to telescope a Bratteli diagram, one takes a subsequence of levels $\{n_k\}$ and considers the set of all finite paths between the new consecutive levels $\{n_k\}$ and $\{n_{k+1}\}$ as edges of the new diagram. In particular, a Bratteli diagram B has rank d if and only if there is a telescoping B' of B such that B' has exactly d vertices at each level. More information about the telescoping procedure can be found in many papers on Bratteli diagrams, for example, in [GPS95].

Lemma 2.3. *Every aperiodic Bratteli diagram B can be telescoped to a diagram B' with the property: $|r^{-1}(v)| \geq 2$, $v \in V \setminus V_0$ and $|s^{-1}(v)| \geq 2$, $v \in V \setminus (V_0 \cup V_1)$.*

In other words, we can state that, for any aperiodic Bratteli diagram, the properties $|r^{-1}(v)| \geq 2$, $v \in V \setminus V_0$, and $|s^{-1}(v)| \geq 2$, $v \in V \setminus (V_0 \cup V_1)$, hold for infinitely many levels n .

Proof. We need to show that for every $n \in \mathbb{N}$ there exists $m > n$ such that for each vertex $v \in V_m$ there are at least two finite paths $e, f \in E(n, m) := E_{n+1} \circ \dots \circ E_m$ with $r(e) = r(f) = v$. Assume that the converse is true. Then there exists n such that for all $m > n$ the set $U_m = \{x = (x_i) \in X_B : |r^{-1}(r(x_i))| = 1, i = n+1, \dots, m\}$ is not empty. Clearly, U_m is a clopen subset of X_B and $U_m \supset U_{m+1}$. It follows that $X_{per} \supset U = \bigcap_{m>n} U_m \neq \emptyset$. This contradicts the aperiodicity of the diagram. \square

Throughout the paper, we consider Bratteli diagrams B for which X_B is a Cantor set and \mathcal{E} is a countable Borel equivalence relation on X_B .

Remark 2.4. Given an aperiodic dynamical system (X, T) , a Bratteli diagram is constructed by a sequence of Kakutani-Rokhlin partitions generated by (X, T) (see [HPS92] and [Me06]). The n -th level of the diagram corresponds to the n -th Kakutani-Rokhlin partition and the number $h_w^{(n)}$ is the height of the T -tower labeled by the symbol w from that partition.

2.2 Orderings on a Bratteli diagram

Let $B = (V, E)$ be a Bratteli diagram whose path space X_B is a Cantor set.

Definition 2.5. A Bratteli diagram $B = (V, E)$ is called *ordered* if a linear order ' $>$ ' is defined on every set $r^{-1}(v)$, $v \in \bigcup_{n \geq 1} V_n$. We use ω to denote the corresponding partial order on B and write (B, ω) when we consider B with the ordering ω . Denote also by \mathcal{O}_B the set of all orderings on B .

Let B be a stationary diagram. We say an ordering $\omega \in \mathcal{O}_B$ is *stationary* if the partial linear order defined by ω on the set E_n of all edges between levels V_{n-1} and V_n , does not depend on n . It is well known that for every stationary ordered Bratteli diagram (B, ω) one can define a ‘substitution τ read on B ’ by the following rule. For each vertex $i \in V = \{1, 2, \dots, d\}$, we write $r^{-1}(i) = \{e_1, \dots, e_t\}$ where $e_1 < e_2 < \dots < e_t$ with respect to ω . Then we set $\tau(i) = j_1 j_2 \dots j_t$ where $j_k = s(e_k)$, $k = 1, \dots, t$; this defines the substitution read on B . Conversely, such a substitution τ describes completely the stationary ordered Bratteli diagram (B, ω) .

Every $\omega \in \mathcal{O}_B$ defines the *lexicographic* ordering on the set $E_{k+1} \circ \dots \circ E_l := \{(e_{k+1}, \dots, e_l) : e_i \in E_i, r(e_i) = s(e_{i+1}), i = k+1, \dots, l-1\}$ of finite paths between vertices of levels V_k and V_l : $(e_{k+1}, \dots, e_l) > (f_{k+1}, \dots, f_l)$ if and only if there is i with $k+1 \leq i \leq l$, $e_i = f_i$ for $i < j \leq l$ and $e_i > f_i$.

It follows that, given $\omega \in \mathcal{O}_B$, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographic ordering generated by ω . We call a finite or infinite path $e = (e_i)$ *maximal (minimal)* if every e_i is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$. Notice that, for $v \in V_i$, $i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\max}(\omega)$ and $X_{\min}(\omega)$ the sets of all maximal and minimal infinite paths from X_B , respectively. It is not hard to show that $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are *non-empty closed subsets* of X_B . Notice that, in general, X_{\max} and X_{\min} may have interior points. For a finite rank Bratteli diagram B , the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are always finite for any ω , and if B has rank d , then each of them have at most d elements (Proposition 6.2 in [BKM09]). An ordered Bratteli diagram (B, ω) is called *properly ordered* if the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are singletons.

Let (B, ω) be an ordered Bratteli diagram. Let (n_k) be a subsequence of levels of B and construct new Bratteli diagram (B', ω') telescoped by telescoping to levels (n_k) . Here ω' is the lexicographic ordering on B' defined by ω . It is not hard to see that

$$|X_{\max}(\omega)| = |X_{\max}(\omega')|, \quad |X_{\min}(\omega)| = |X_{\min}(\omega')|.$$

Let $B = (V, E, \omega)$ be an ordered Bratteli diagram. Then $x \in X_{\max}(\omega) \cap X_{\min}(\omega)$ if and only if $|\mathcal{E}(x)| = 1$. Thus, if B is an aperiodic Bratteli diagram, then $X_{\max}(\omega) \cap X_{\min}(\omega) = \emptyset$.

Now, we give a useful description of infinite paths in an ordered Bratteli diagram $B = (V, E, \omega)$ (see also [BDK06]). Take $v \in V_n$ and consider the finite set $E(v_0, v)$ whose cardinality is $h_v^{(n)}$. The lexicographic ordering on $E(v_0, v)$ gives us an enumeration of its elements from 0 to $h_v^{(n)} - 1$ where 0 is assigned to the minimal path and $h_v^{(n)} - 1$ is assigned to the maximal path in $E(v_0, v)$. Recall that we assume

$h_v^{(1)} = 1$, $v \in V_1$, and we have by induction

$$h_v^{(n)} = \sum_{w \in s(r^{-1}(v))} |E(w, v)| h_w^{(n-1)}, \quad v \in V_n.$$

Let $y = (e_1, e_2, \dots)$ be an infinite path from X_B . Consider a sequence (P_n) of enlarging finite paths defined by y where $P_n = (e_1, \dots, e_n) \in E(v_0, r(e_n))$, $n \in \mathbb{N}$. Then every P_n can be identified with a pair (i_n, v_n) where $v_n = r(e_n)$ and $i_n \in [0, h_{v_n}^{(n)} - 1]$ is the number assigned to P_n in $E(v_0, v_n)$. Thus, every $y = (e_n) \in X_B$ is uniquely represented as the infinite sequence (i_n, v_n) with $v_n = r(e_n)$ and $0 \leq i_n \leq h_{v_n}^{(n)} - 1$. We refer to the sequence (i_n, v_n) as the *associated sequence*.

Proposition 2.6. *Two infinite paths $e = (e_1, e_2, \dots)$ and $e' = (e'_1, e'_2, \dots)$ are cofinal with respect to \mathcal{E} if and only if the sequences (i_n, v_n) and (i'_n, v'_n) associated to e and e' satisfy the condition: there exists $m \in \mathbb{N}$ such that $v_n = v'_n$ and $i_n - i'_n = i_m - i'_m$ for all $n \geq m$.*

Proof. Suppose e and e' are cofinal. Take m such that $e_n = e'_n$ for all $n \geq m$. Consider the associated sequences (i_n, v_n) and (i'_n, v'_n) . Then we see that $v_n = v'_n$ for all $n \geq m$. Without loss of generality, we can assume that $c_m = i_m - i'_m \geq 0$. This means that the finite path $P_m = P(e_1, \dots, e_m)$ is the c_m -th successor of the finite path $P'_m = P(e'_1, \dots, e'_m)$. Let $c_{m+1} = i_{m+1} - i'_{m+1}$. By definition of the lexicographic ordering on $E(v_0, v_{m+1})$, we obtain that $c_{m+1} = c_m$. Thus, by induction, $c_n = c_m$ for all $n \geq m$.

Conversely, suppose that two associated sequences (i_n, v_n) and (i'_n, v'_n) possess the property: there exists $m \in \mathbb{N}$ such that $v_n = v'_n$ and $i_n - i'_n = i_m - i'_m$ for all $n \geq m$. To see that e and e' are cofinal, notice that e_{m+1} and e'_{m+1} are in $E(v_m, v_{m+1})$. By definition of the lexicographic ordering on $E(v_0, v_{m+1})$, we conclude that $e_{m+1} = e'_{m+1}$. \square

Proposition 2.7. *A Bratteli diagram $B = (V, E)$ admits an ordering $\omega \in \mathcal{O}_B$ on B with $\text{Int}(X_{\max}(\omega)) \neq \emptyset$ if and only if there exist $x = (x_i) \in X_B$ and $n > 0$ such that $U(x_1, \dots, x_n) = \{y \in X_B : y_i = x_i, i = 1, \dots, n\}$ has no cofinal paths, i.e. $U(x_1, \dots, x_n)$ meets each \mathcal{E} -orbit at most once. A similar result holds for $\text{Int}(X_{\min}(\omega))$.*

Proof. Let x be an interior point of $X_{\max}(\omega)$. Then there is $n > 0$ such that $U(x_1, \dots, x_n) \subset X_{\max}(\omega)$. Therefore, $U(x_1, \dots, x_n)$ contains no distinct cofinal paths.

Now, suppose that there exist $x = (x_i) \in X_B$ and $n > 0$ such that $U = U(x_1, \dots, x_n)$ meets each \mathcal{E} -orbit at most once. Define a linear order ω_v on $r^{-1}(v)$, $v \in V \setminus V_0$, as follows. If there exists $e \in r^{-1}(v)$ which is an edge in

an infinite path $y \in U$, then we order $r^{-1}(v)$ such that e is maximal in $r^{-1}(v)$. If such an e does not exist, we order $r^{-1}(v)$ in an arbitrary way. It follows that for this ordering $U \subset X_{\max}(\omega)$. \square

A Bratteli diagram $B = (V, E)$ is called *regular* if for any ordering $\omega \in \mathcal{O}_B$ the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have empty interior. In particular, finite rank Bratteli diagrams are regular.

Given a Bratteli diagram B , we can easily describe the set of all orderings \mathcal{O}_B in the following way. To every vertex $v \in V$ we assign the finite set $P_v = \{1, \dots, |r^{-1}(v)|!\}$. Then \mathcal{O}_B is represented as follows:

$$\mathcal{O}_B = \prod_{v \in V} P_v = \{v_0\} \times \prod_{v \in V_1} P_v \times \dots \times \prod_{v \in V_i} P_v \times \dots \quad (2.1)$$

It follows from (2.1) that \mathcal{O}_B is a Cantor set with respect to the product topology. In other words, two orderings $\omega = (\omega_v)$ and $\omega' = (\omega'_v)$ from \mathcal{O}_B are close if they agree on a sufficiently long initial segment: $\omega_v = \omega'_v, v \in \bigcup_{i=0}^k V_i$.

It is worth noticing that the order space \mathcal{O}_B is sensitive with respect to a telescoping. Indeed, let B be a Bratteli diagram and B' denote the diagram obtained by telescoping of B with respect to a subsequence (n_k) of levels. We see that any ordering ω on B can be extended to the (lexicographic) ordering ω' on B' . Hence the map $L : \omega \rightarrow \omega' = L(\omega)$ defines a closed proper subset $L(\mathcal{O}_B)$ of $\mathcal{O}_{B'}$.

The set of all orderings \mathcal{O}_B on a Bratteli diagram $B = (V, E)$ can be considered also as a *measure space* whose Borel structure is generated by cylinder sets. On the set

$$\mathcal{O}_B = \prod_{v \in V} \{1, \dots, |r^{-1}(v)|!\},$$

we take the product measure $\mu = \prod_{v \in V} \nu_v$ where ν_v is a measure on the set $Y_v = \{1, \dots, |r^{-1}(v)|!\}$. The case of the uniformly distributed measure μ_v on Y_v is of particular interest: $\mu_v(\{i\}) = (|r^{-1}(v)|!)^{-1}$ for every $i \in Y_v$ and $v \in V$ and μ is the product of μ_v 's.

Let $\omega = (\omega_v)_{v \in V} \in \mathcal{O}_B$ be an ordering on B where ω_v is a linear order on $r^{-1}(v)$. Then ω_v can be identified with a permutation ρ_v on the set $\{1, \dots, |r^{-1}(v)|\}$, $v \in V$. Indeed, let ω_0 be the ordering on B (called *natural*) such that $\omega_v(0)$ is the left-to-right order on $r^{-1}(v)$. Then ω_v determines a permutation ρ_v of the natural order. Conversely, any family of permutations (ρ_v) defines an ordering on B .

2.3 Vershik map

Definition 2.8. Let $B = (V, E, \omega)$ be an ordered Bratteli diagram. We say that $\varphi = \varphi_\omega : X_B \rightarrow X_B$ is a (*continuous*) *Vershik map* if it satisfies the following

conditions:

- (i) φ is a homeomorphism of the Cantor set X_B ;
- (ii) $\varphi(X_{\max}(\omega)) = X_{\min}(\omega)$;
- (iii) if an infinite path $x = (x_1, x_2, \dots)$ is not in $X_{\max}(\omega)$, then $\varphi(x_1, x_2, \dots) = (x_1^0, \dots, x_{k-1}^0, \overline{x_k}, x_{k+1}, x_{k+2}, \dots)$, where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of x_k in $r^{-1}(r(x_k))$, and $(x_1^0, \dots, x_{k-1}^0)$ is the minimal path in $E(v_0, s(\overline{x_k}))$.

If ω is an ordering on B , then one can always define the map φ_0 that maps $X_B \setminus X_{\max}(\omega)$ onto $X_B \setminus X_{\min}(\omega)$ according to (iii) of Definition 2.8. The question about the existence of the Vershik map is equivalent to that of an extension of $\varphi_0 : X_B \setminus X_{\max}(\omega) \rightarrow X_B \setminus X_{\min}(\omega)$ to a homeomorphism of the entire set X_B . If ω is a proper ordering, then φ_ω is a homeomorphism. For a finite rank Bratteli diagram B , the situation is simpler than for a general Bratteli diagram because the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are finite.

Definition 2.9. Let B be a Bratteli diagram B . We say that an ordering $\omega \in \mathcal{O}_B$ is *perfect* if ω generates a Vershik map φ_ω on X_B . Denote by \mathcal{P}_B the set of all *perfect* orderings on B . We call an ordering belonging to \mathcal{P}_B^c (the complement of \mathcal{P}_B in \mathcal{O}_B) *imperfect*.

We observe that for a regular Bratteli diagram with an ordering ω , the Vershik map φ_ω , if it exists, is defined in a unique way. More precisely, if B is a regular Bratteli diagram such that the set \mathcal{P}_B is not empty, then the map $\Phi : \omega \mapsto \varphi_\omega : \mathcal{P}_B \rightarrow \text{Homeo}(X_B)$ is injective. Also, a necessary condition for $\omega \in \mathcal{P}_B$ is that $|X_{\max}(\omega)| = |X_{\min}(\omega)|$.

Remark 2.10. We first note that if B is a simple Bratteli diagram, then the set $\mathcal{P}_B \neq \emptyset$. Indeed, it is not hard to see that if x and y are two paths from X_B going through disjoint edges at each level, then one can find an ordering ω on B such that $X_{\max}(\omega) = \{x\}$ and $X_{\min}(\omega) = \{y\}$, so that (B, ω) is properly ordered, and $\omega \in \mathcal{P}_B$. Another example of a perfect order for a simple finite rank Bratteli diagram is the following: for each $v \in V$, let ω be the natural order (from left-to-right) on $r^{-1}(v)$.

In the next section, we will describe a class of non-simple Bratteli diagrams that do not admit a perfect ordering.

Proposition 2.11. *Given a simple Bratteli diagram $B = (V, E)$, the condition $\mathcal{P}_B = \mathcal{O}_B$ holds if and only if $|V_n| = 1$ for infinitely many levels n .*

Proof. The part ‘if’ is obvious because the condition $|V_n| = 1$ for infinitely many levels n implies any ordering is proper.

Conversely, suppose that there is some level K such that $|V_n| \geq 2$ for all levels $n > K$ and $|V_K| = 1$. We will construct an imperfect ordering ω on B . Fix two vertices v and v' at each level V_n , $n > K$, independent of n . By Lemma 2.3, we can choose two infinite vertical path, x and y , going through the vertex v for all levels $n > K$. Choose an order ω on $r^{-1}(v)$ such that x is a minimal path and y is a maximal path. For the vertex v' we define an order on $r^{-1}(v')$ such that a vertical path z going through v' is maximal. For all other vertices $w \in V_n$, $n > K$, the order ω is defined such that no more maximal and minimal paths exist on B . To do this, it suffices to enumerate edges in $r^{-1}(w)$ such that the source of any maximal edge was either v or v' , and the source of any minimal edge in $r^{-1}(w)$ was v . (Here we assume that all entries of incidence matrices are non-zero. In the general case, one has to take finite paths with the same property). By construction, ω has two maximal paths and one minimal path, so that $\omega \notin \mathcal{P}_B$. \square

In contrast, one can find aperiodic diagrams for which any ordering is perfect. Indeed, it suffices to take a rooted tree and turn it into a non-simple Bratteli diagram by replacing every single edge with a strictly larger number of edges. Then such a diagram consists of uncountably many odometers and clearly every ordering on it produces a continuous Vershik map.

Remark 2.12. Let (B, ω) be an ordered Bratteli diagram and let ω' be an ordering on B such that ω and ω' are different on $r^{-1}(v)$ only for a finite number of vertices v . Then ω is perfect if and only if ω' is perfect. Similarly, we note that if B and B' are Kakutani equivalent Bratteli diagrams, then $\mathcal{P}_B = \mathcal{P}_{B'}$.

Proposition 2.13. *Let B be a regular Bratteli diagram such that the set \mathcal{P}_B is not empty. Let \mathcal{P}_B be equipped with the topology induced from \mathcal{O}_B and let the set $\Phi(\mathcal{P}_B)$ be equipped with the topology of uniform convergence induced from the group $\text{Homeo}(X_B)$ where the map $\Phi : \omega \mapsto \varphi_\omega$ has been defined above. Then $\Phi : \mathcal{P}_B \rightarrow \Phi(\mathcal{P}_B)$ is a homeomorphism.*

Proof. We need only to show that Φ and Φ^{-1} are continuous because injectivity of Φ is obvious.

Fix an ordering $\omega_0 \in \mathcal{P}_B$ and let φ_{ω_0} be the corresponding Vershik map. Consider a neighborhood $W = W(\varphi_{\omega_0}; E_1, \dots, E_k) = \{f \in \text{Homeo}(X_B) : f(E_i) = \varphi_{\omega_0}(E_i), i = 1, \dots, k\}$ of φ_{ω_0} defined by clopen sets E_1, \dots, E_k . Take $m \in \mathbb{N}$ such that all clopen sets E_1, \dots, E_k ‘can be seen’ at the first m levels of the diagram B . This means that every set E_i is a finite union of the cylinder sets defined by finite paths of length m .

Suppose $\omega_n \rightarrow \omega_0$ where $\omega_n \in \mathcal{P}_B$. By (2.1), the ordering ω_0 is an infinite path in the product $\prod_{v \in V} P_v$. Let Q be the neighborhood of ω_0 in \mathcal{O}_B which is defined by the finite part of ω_0 from v_0 to V_{m+1} . Find N such that $\omega_n \in Q$ for all $n \geq N$.

This means that the ordering ω_n ($n \geq N$) agrees with ω_0 on the first $m+1$ levels of the diagram B . Therefore, φ_{ω_n} acts as φ_{ω_0} on all finite paths from v_0 to V_m . Hence, $\varphi_{\omega_n}(E_i) = \varphi_{\omega_0}(E_i)$ and $\varphi_{\omega_n} \in W$.

Conversely, let $\varphi_{\omega_n} \rightarrow \varphi_\omega$ in the topology of uniform convergence; we prove that $\omega_n \rightarrow \omega$. Take a neighborhood $Q(\omega)$ which consists of all orderings ω' such that ω' agrees with ω on the sets $r^{-1}(v)$ where $v \in \bigcup_{i=1}^N V_i$. Let F_1, \dots, F_p denote all cylinder subsets of X_B corresponding to the finite paths between v_0 and the vertices from $\bigcup_{i=1}^{N+1} V_i$. Consider the neighborhood $W = W(\varphi_\omega; F_1, \dots, F_p)$. Then there exists $m \in \mathbb{N}$ such that $\varphi_{\omega_i} \in W$ for $i \geq m$. This means that $\varphi_{\omega_i}(F_j) = \varphi_\omega(F_j)$ for all $j = 1, \dots, p$. Let us check that $\omega_i \in Q(\omega)$ for $i \geq m$. Indeed, if one assumes that $\omega' \notin Q(\omega)$ then there exists the least k and a vertex $v \in V_k$ such that ω and ω' define different linear orders on $r^{-1}(v)$ but ω and ω' agree for all $v \in \bigcup_{i=1}^{k-1} V_i$. Let e be an edge from $r^{-1}(v)$ such that the ω -successor and ω' -successor of e are different edges. Then take the cylinder set F which corresponds to the finite path (f, e) where f is the the maximal path from v_0 to $s(e)$ for the both orders. It follows from the above construction that $\varphi_\omega(F) \neq \varphi_{\omega'}(F)$, a contradiction. \square

Let $\omega = (\omega_v)_{v \in V}$ be an ordering on a regular Bratteli diagram $B = (V, E)$. For every $x_{\max} = (x_n) \in X_{\max}(\omega)$, we define the set $Succ(x_{\max}) \subset X_{\min}(\omega)$ as follows: $y_{\min} = (y_n)$ belongs to the set $Succ(x_{\max})$ if for infinitely many n there exist edges $y'_n \in s^{-1}(r(x_n))$ and $y''_n \in s^{-1}(r(y_n))$ such that $r(y'_n) = r(y''_n) = v_n$ and y''_n is the successor of y'_n in the set $r^{-1}(v_n)$. Given a path $y_{\min} \in X_{\min}(\omega)$, we define the set $Pred(y_{\min}) \subset X_{\max}(\omega)$ in a similar way. It is not hard to prove that the sets $Succ(x_{\max})$ and $Pred(y_{\min})$ are non-empty and closed for any x_{\max} and y_{\min} .

Theorem 2.14. *Let B be a simple Bratteli diagram with $|V_n| \geq 2$ for all levels n . Then both sets \mathcal{P}_B and \mathcal{P}_B^c are dense in \mathcal{O}_B . Moreover, $\text{int}(\mathcal{P}_B) = \text{int}(\mathcal{P}_B^c) = \emptyset$.*

Proof. Since B is simple, the set \mathcal{P}_B is not empty. By Proposition 2.11, $\mathcal{P}_B^c \neq \emptyset$. Take an ordering $\omega \in \mathcal{O}_B$ and consider its neighborhood $U_N(\omega) = \{\omega' \in \mathcal{O}_B : \omega \text{ and } \omega' \text{ coincide on } r^{-1}(v) \text{ for all } v \in \bigcup_{i=1}^N V_i\}$. Then there exists a perfect ordering ω_g belonging to $U_N(\omega)$. To see this, it suffices to make the diagram B properly ordered, i.e., with single maximal and minimal paths. Choose a vertex $v \in V_N$ and let $x = (x_i)$ and $y = (y_i)$ be two infinite paths such that their initial segments (x_1, \dots, x_N) and (y_1, \dots, y_N) are the minimal and maximal finite path in $E(v_0, v)$ with respect to ω . Define ω_g for levels V_i ($i > N$) so that x becomes the unique minimal path and y becomes the unique maximal path (see Proposition 2.11 for details).

Conversely, we assume that ω is perfect and we will show that any neighborhood of ω contains an imperfect ordering ω_b . We can do this, for instance, by constructing ω_b so that $|X_{\max}(\omega_b)| \neq |X_{\min}(\omega_b)|$. We choose a sufficiently large level N and take a new order ω' on $r^{-1}(v)$, $v \in \bigcup_{i > N} V_i$ such that for a maximal path $x \in X_{\max}(\omega')$

the set $Succ(x) \subset X_{\min}(\omega')$ contains at least two points. Hence, the ordering ω_b that agrees with ω until level N and then coincides with ω' past level N is imperfect. \square

3 Finite rank ordered Bratteli diagrams

In this section, we focus on the study of orderings on a finite rank Bratteli diagram B . To do this, we define new notions related to a finite rank Bratteli diagram that will be used in our considerations. These are the *skeleton*, *associated graph*, and *language* of B . We find necessary and sufficient conditions on an ordering ω so that it is perfect. We explicitly describe the class of diagrams of finite rank d that can have a perfect ordering with exactly $k \leq d$ maximal and minimal paths. It turns out that the number of extremal paths determines some constraints on the entries of incidence matrices; this is most apparent when $k = d$. We also find a class of non-simple diagrams that do not admit any perfect ordering.

3.1 Language of a finite rank diagram

Let ω be an ordering on a finite rank Bratteli diagram B . We define the notion of a *language* generated by ω on B . For this we assume that $V_n = V$ for each n , where $|V| = d$. Fix a vertex $v \in V_n$ and some $m < n$, consider $\bigcup_{w \in V_m} E(w, v)$ as the ω -ordered set $\{e_1, \dots, e_p\}$ where $e_i < e_{i+1}$ for $1 \leq i \leq p-1$. Define the word $w(v, m, n) := s(e_1)s(e_2)\dots s(e_p)$ over the alphabet V . We use the notation $w' \subseteq w$ to indicate that w' is a subword of w .

Definition 3.1. The set

$$\mathcal{L}_{B,\omega} = \{w : w \subseteq w(v_n, m_n, n), v_n \in V_n, m_n < n, n \in \mathbb{N} \text{ for infinitely many } n\}$$

is called the *language* of B with respect to the ordering ω .

We remark that the notion of the language $\mathcal{L}_{B,\omega}$ is not always robust under telescoping: let (B', ω') be a telescoping of an ordered Bratteli diagram (B, ω) where ω' is the lexicographic ordering defined by ω . Then $\mathcal{L}_{B',\omega'} \subset \mathcal{L}_{B,\omega}$ where the inclusion can be strict. For example, consider B where

$$F_{2n} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad F_{2n-1} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, \quad n \geq 1. \quad (3.2)$$

Let ω be defined by the substitution $\tau_1(a) = aba$, $\tau_1(b) = aaba$ on E_{2n} , and by the substitution $\tau_2(a) = bab$, $\tau_2(b) = abba$ on E_{2n-1} for $n \geq 1$. Then $\{aa, ab, ba, bb\} \subset \mathcal{L}_{B,\omega}$. Now telescope B to every second level to get the stationary Bratteli diagram B' whose incidence matrix is

$$F'_n = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 10 & 4 \end{pmatrix} \quad (3.3)$$

for each $n \geq 1$, so that $\omega' := L(\omega)$ is defined by the substitution $\tau := \tau_1 \circ \tau_2$ where $\tau(a) = aaba\ aba\ aaba$ and $\tau(b) = aba\ aaba\ aaba\ aba$, then $bb \notin \mathcal{L}_{B',\omega'}$. Note however that both ω and ω' are perfect, in accordance with Lemma 3.3 below.

Also, in the special case where B is stationary and ω is defined by a substitution τ (so that ω is also stationary), we see that $\mathcal{L}_{B,\omega}$ is precisely the language \mathcal{L}_τ defined by the substitution τ , and in this case, if B' is a telescoping of B to levels (n_k) with $\omega' = L(\omega)$, then $\mathcal{L}_{B,\omega} = \mathcal{L}_{B',\omega'}$. Indeed, any word $w \in \mathcal{L}_{B,\omega}$ is a subword of $\tau^n(a)$ for some $n \in \mathbb{N}$ and letter a . Now the order on the k -th level of B' is generated by $\tau^{n_k - n_{k-1}}$ and as long as $n_K - n_k > n$, we will see w as a subword of $w(a, n_k, n_K) \subset \mathcal{L}_{B',\omega'}$. The relationship between $\mathcal{L}_{B,\omega}$ and the continuity of the Vershik map has been studied in [Yas11] in the case where ω is stationary, i.e., generated by a substitution, and also in [HZ01]¹.

In the special case where B has a constant number of vertices at each level, and a maximal (minimal) path M (m) goes through the same vertex v_M (v_m) at each level of B , we will call this path *vertical*.

The following proposition characterizes when ω is a perfect ordering on a finite rank Bratteli diagram. It is evident that Part (2) of Proposition 3.2 is the more general statement; we use it mainly to prove Lemma 3.3, which allows us to telescope an ordered diagram without losing (or gaining) perfection, and assume, if needed, that we are in the special case described by Part (1). We also use Part (2) of Proposition 3.2 in the case where we want to make measure theoretic statements about sets in (\mathcal{O}_B, μ) ; for if B' is a nontrivial telescoping of B , the set $L(\mathcal{P}_B)$ is a set of measure 0 in $\mathcal{P}_{B'}$.

Proposition 3.2. *Let (B, ω) be a finite rank ordered Bratteli diagram.*

1. *Suppose $|V_n| = d$ for each $n \in \mathbb{N}$, and that the ω -maximal and ω -minimal paths M_1, \dots, M_k and $m_1, \dots, m_{k'}$ are vertical passing through the vertices v_{M_1}, \dots, v_{M_k} and $v_{m_1}, \dots, v_{m_{k'}}$ respectively. Then ω is perfect if and only if*

- (a) $k = k'$,
- (b) *there is a permutation σ of $\{1, \dots, k\}$ such that for each $i \in \{1, \dots, k\}$, $v_{M_i} v_{m_j} \in \mathcal{L}_{B,\omega}$ if and only if $j = \sigma(i)$.*

2. *Let M be a maximal path, and m, m' be distinct minimal paths with respect to ω (we do not assume here that these paths are vertical). Then the following statements are equivalent:*

¹The relevant formula on Page 5 is incorrect in the final version: the correct version is in the preprint which can be found at <http://combinatorics.cis.strath.ac.uk/papers/lucasz>.

- (a) $\omega \notin \mathcal{P}_B$;
- (b) there exist strictly increasing sequences of levels (n_k) , (n_k^*) , (N_k) and (N_k^*) , vertices $\{w_k, v_k\} \subset V_{n_k}$, $\{w_k^*, v_k^*\} \subset V_{n_k^*}$, vertices $u_k \in V_{N_k}$, $u_k^* \in V_{N_k^*}$ such that M passes through w_k and w_k^* , m and m^* pass through v_k and v_k^* respectively, and $w_k v_k \subset w(u_k, n_k, N_k)$, $w_k^* v_k^* \subset w(u_k^*, n_k^*, N_k^*)$.

Proof. We first assume that the Vershik map φ_ω exists. Then φ_ω generates a one-to-one map between the finite sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ by sending each M_i to some m_j : let $\sigma(i) = j$. Clearly, $k = k'$. We need to check that $v_{M_i} v_{m_j}$ is in the language $\mathcal{L}_{B,\omega}$ if and only if $j = \sigma(i)$. It follows from continuity of φ_ω and the relation $\varphi_\omega(M_i) = m_j$ that if $x_n \rightarrow M_i$ then $\varphi_\omega(x_n) = y_n \rightarrow m_j$ as $n \rightarrow \infty$. We see that, for every n , the condition $\varphi_\omega(x_n) = y_n$ implies that $v_{M_i} v_{m_j} \in w(v, m, N)$ for some $v \in V_N$ and some $m < N$, because x_n and y_n are taken from neighborhoods generated by finite paths going through v_{M_i} and v_{m_j} respectively. Furthermore, as $n \rightarrow \infty$, so does N and also m . Hence $v_{M_i} v_{m_j} \in \mathcal{L}_{B,\omega}$ when $j = \sigma(i)$. It is clear that if $v_{M_i} v_{m_k} \in \mathcal{L}_{B,\omega}$ for some $k \neq \sigma(i)$, then φ_ω would send M_i also to m_k , a contradiction.

Conversely, assuming that (1a) and (1b) hold, extend φ_ω to $X_{\max}(\omega)$ by defining $\varphi(M_i) := m_{\sigma(i)}$. It is obvious that φ_ω is one-to-one. Fix a pair (M_i, m_j) where $j \neq \sigma(i)$, and let $x_n \rightarrow M_i$ as $n \rightarrow \infty$; we show that $y_n = \varphi_\omega(x_n) \rightarrow m_j$. Telescoping the diagram, we can assume that the first n edges of x_n coincide with those of M_i , i.e. $x_n = \overline{e}_{\max}^{(n)}(v_0, v_{M_i}) e_{n+1} e_{n+2} \cdots$ where e_{n+1} is not maximal in $r^{-1}(r(e_{n+1}))$. Then $y_n = \overline{f}_{\min}^{(n)}(v_0, s(e'_{n+1})) e'_{n+1} e_{n+2} \cdots$ where e'_{n+1} is the successor of e_{n+1} . Take a subsequence (y'_n) of (y_n) convergent to a point $z \in X_B$. By construction, z must be a minimal path. It follows from the uniqueness of j in condition (1b) that $z = m_j$; this proves the continuity of φ_ω .

The proof of Part (2) is similar to that of Part (1), so we omit it (see also Proposition 3.4 below), although Figure 1 is explanatory.

□

The next lemma allows us to move the ordering ω on B to the lexicographical ordering $L(\omega)$ on a telescoping B' of B without worrying about altering perfection: in particular we can telescope (B, ω) in order to have the following property: $|V_n| = d$ for each n , and the maximal paths M_1, \dots, M_k and minimal paths $m_1, \dots, m_{k'}$ are vertical in the diagram (B', ω') , so that the criteria in Part 1 of Proposition 3.2 for ω to belong to \mathcal{P}_B can be verified.

Lemma 3.3. *Let B be a Bratteli diagram of finite rank and B' a telescoping of B . Then an ordering $\omega \in \mathcal{P}_B$ if and only if the corresponding lexicographic ordering $\omega' = L(\omega) \in \mathcal{P}_{B'}$.*

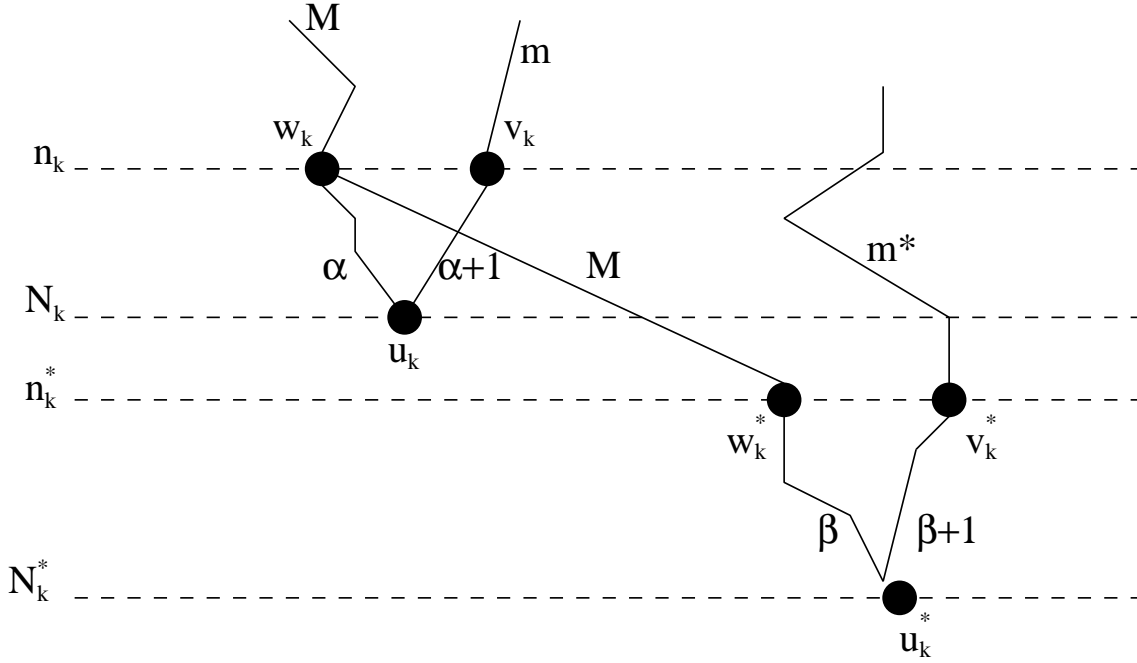


Figure 1: A discontinuous ϕ_ω .

Proof. If ω does not determine a Vershik map, then by Part (2) of Proposition 3.2, there is a maximal path M , two distinct minimal paths m and m^* , infinite sequences of levels (n_k) and (n_k^*) , (N_k) and (N_k^*) , vertices $\{w_k, v_k\} \subset V_{n_k}$, $\{w_k^*, v_k^*\} \subset V_{n_k^*}$ and vertices $u_k \in V_{N_k}$, $u_k^* \in V_{N_k^*}$ such that M passes through w_k and w_k^* , m (m^*) pass through v_k (v_k^*), and $w_k v_k \subset w(u_k, n_k, N_k)$, $w_k^* v_k^* \subset w(u_k^*, n_k^*, N_k^*)$. If the images of M , m and m^* in B' are denoted by M' , m' and $(m^*)'$ respectively, then the paths m' and $(m^*)'$ are distinct. Note that in B , it cannot be the case that for infinitely many levels, the minimal paths go through the same vertex - otherwise they are not distinct. Thus, there is some N such that if $n \geq N$, the level n edge in m has a different source and range from the level n edge in m^* .

Let B' be a telescoping of B . Find the levels m_j and M_j such that $m_{j-1} < n_k \leq m_j$, $M_{j-1} < N_k \leq M_j$, and let E'_j denote the edge set in B' obtained by telescoping between m_{j-1} -st and m_j -th levels of B . Similarly, E'_j is obtained by telescoping between the M_{j-1} -st and M_j -th levels of B . Let the path M go through $w'_j \in V_{m_j}$, and m through $v'_j \in V_{m_j}$. Let $u'_j \in V_{M_j}$ be any vertex such that there is a path from $u_k \in V_{N_k}$ to u' . Then for the corresponding vertices $w'_{j-1}, v'_{j-1} \in V'_{j-1}$ and $u'_j \in V'_j$ respectively it is the case that $w'_{j-1} v'_{j-1} \in w(u'_j, j-1, J)$, with M' passing through w'_{j-1} , and m' passing through v'_{j-1} . Pass to a subsequence of $((w'_{j-1}, v'_{j-1}, u'_j))$, and repeat this procedure for m^* . Since all but a finite initial part of m and m^* live on disjoint vertices, these two subsequences are distinct. By Part (2) of Proposition

3.2, the ordering ω' on B' , obtained from ω by telescoping, does not determine a Vershik map.

Conversely, suppose that ω' does not determine a Vershik map. Then Proposition 3.2 applies to ω' , yielding the sequences and vertices with the specified properties; via the process of splitting B' to get back B , these sequences and vertices in B' are mapped to sequences and vertices in B with the properties required by Proposition 3.2, implying that $\omega \in \mathcal{P}^c$. \square

Now we give another criterion which guarantees the existence of Vershik map on an ordered Bratteli diagram $B = (V, E, \omega)$ (not necessarily of finite rank).

Proposition 3.4. *An ordering $\omega = (\omega_v)_{v \in V}$ on a regular Bratteli diagram B is perfect if and only if for every $x_{\max} \in X_{\max}(\omega)$ and $y_{\min} \in X_{\min}(\omega)$ the sets $\text{Succ}(x_{\max})$ and $\text{Pred}(y_{\min})$ are singletons.*

Proof. We first observe that, if $\text{Succ}(x_{\max}) = \{y_{\min}\}$, then one can define $\varphi_\omega : x_{\max} \rightarrow y_{\min}$ where x_{\max} is any path from $X_{\max}(\omega)$. Since $\text{Pred}(y_{\min})$ is also a singleton, we obtain a one-to-one correspondence between the sets of maximal and minimal paths. The fact that φ_ω is continuous can be checked directly.

If ω is perfect, then it follows from the existence of the Vershik map φ_ω that either of the sets $\text{Succ}(x_{\max})$ and $\text{Pred}(y_{\min})$ must be singletons. \square

Let $\mathcal{L} \subset \mathcal{A}^\mathbb{N}$. A word $W \in \mathcal{L}$ is *periodic* if it can be written as a concatenation $W = U^k$ of k copies of a word U where $k > 1$. Given a word $W = w_1 \dots w_p$, we define $\sigma^i(W) := w_{i+1}w_{i+2} \dots w_p w_1 \dots w_i$. We say that \mathcal{L} is *periodic* if there is some word $V \in \mathcal{L}$ such that any word $W \in \mathcal{L}$ is of the form SV^kP for some suffix (prefix) $S = S(V)$ ($P = P(V)$) of V . Finally if $\mathcal{Q} = \{q_1, q_2, \dots, q_n\}$ is a partition of X and $T : X \rightarrow X$, we say that \mathcal{Q} is *periodic* for T if $T(q_i) = q_{i+1}$ for $1 \leq i < n$ and $T(q_n) = q_1$.

Next we state and prove a result which Fabien Durand has communicated to us as a known result; the proof below is a direct generalisation of the proof of Part (ii) of Proposition 16 in [DHS99].

Proposition 3.5. *Let ω be a perfect ordering on B . If $\mathcal{L}_{B,\omega}$ is periodic, then (X_B, φ_ω) is topologically conjugate to an odometer.*

Proof. Suppose $\mathcal{L}_{B,\omega}$ is periodic. Fix \bar{v} such that there is a vertical minimal path going through the vertex \bar{v} . (We may have to telescope (B, ω) to obtain this but the resulting diagram also has a periodic language.) Then for all k , $\lim_{n \rightarrow \infty} w(\bar{v}, k, n)$ exists. In particular $\lim_{n \rightarrow \infty} w(\bar{v}, 1, n) = WWW \dots$ where $W = w_1 w_2 \dots w_p$ is of length p and is not periodic. Note that for each $v \in V$ and each $n \geq 2$, $w(v, n-1, n) = S_v^{(n)} W \alpha_v^{(n)} P_v^{(n)}$ with $S_v^{(n)}$ a proper suffix of W , $P_v^{(n)}$ a proper prefix of W , and, whenever $vw \in \mathcal{L}(B, \omega)$, then $P_v^{(n)} S_w^{(n)}$ is either empty or equal to W .

We define partitions $(\mathcal{Q}^{(n)})$ that will be refining, clopen, generating periodic partitions of (X_B, φ_ω) , and such that $|\mathcal{Q}_{n+1}|$ is a multiple of $|\mathcal{Q}_n|$. The existence of this sequence implies that (X_B, φ_ω) is an odometer. For $x = x_1 x_2 \dots \in X_B$, where $s(x_1) = v_0$, $j \in \mathbb{N}$ and $0 \leq i \leq p-1$, let

$$[i]_j = \{x : s(x_{j+1}) s((\phi_\omega(x))_{j+1}) \dots s((\phi_\omega^{p-1}(x))_{j+1}) = \sigma^i(W)\}.$$

Let

$$\mathcal{Q}_1 := \{[i]_1 : 0 \leq i \leq p-1\}$$

Since W is not periodic each x lives in only one $[i]_1$; and \mathcal{Q}_1 is of period p for φ_ω .

Given a vertex $v \in V_n$, recall that $h_v^{(n)} = |E(v_0, v)| = |w(v, 1, n)| =$ for $v \in V_n$. (Here we can assume that there are single edges from v_0 to vertices in V_1 .) Define for $n > 1$

$$\mathcal{Q}_n := \{[i_1, i_2] : 0 \leq i_2 \leq p-1, 0 \leq i_1 \leq l_{w_{i_2+1}}^n - 1, \}$$

where

$$[i_1, i_2] := [i_2]_n \cap \{x : x_1 x_2 \dots x_n \in E(v_0, w_{i_2+1}) \text{ and has } \omega\text{-label } i_1 \}.$$

Then for each n , \mathcal{Q}_n is a clopen partition, \mathcal{Q}_{n+1} refines \mathcal{Q}_n , and it is clear that (\mathcal{Q}_n) is a generating sequence of partitions. We claim that \mathcal{Q}_n is ϕ_ω periodic. For, if $i_1 < h_{w_{i_2+1}}^{(n)} - 1$, $\phi_\omega([i_1, i_2]) = ([i_1 + 1, i_2])$. If $i_1 = h_{w_{i_2+1}}^{(n)} - 1$ and $i_2 < p-1$ then $\phi_\omega([i_1, i_2]) = [(0, i_2 + 1)]$. Finally $\phi_\omega([h_{w_{i_2+1}}^{(n)} - 1, p-1]) = [0, 0]$.

It remains to show that $|\mathcal{Q}_{n+1}|$ is a multiple of $|\mathcal{Q}_n|$. If $W' \subset W$, let $\#_{W'}(W)$ denote the distinct number of occurrences of W' in W . Note that $w_p \overline{w}_1 \in \mathcal{L}(B, \omega)$, so that for each n , $P_{w_p}^{(n)} S_{w_1}^{(n)} = W$ or is the empty word. This means that the following identity is true:

$$|\mathcal{Q}_{n+1}| = \begin{cases} |\mathcal{Q}_n| \sum_{v \in V} \#_v(W) \alpha_v^{(n+2)} |W| + \sum_{v,w} \#_{vw}(W) |P_v^{(n+2)} S_w^{(n+2)}| + 1 & \text{if } P_{w_p}^{(n+2)} S_{w_1}^{(n+2)} = W \\ |\mathcal{Q}_n| \sum_{v \in V} \#_v(W) \alpha_v^{(n+2)} |W| + \sum_{v,w} \#_{vw}(W) |P_v^{(n+2)} S_w^{(n+2)}| & \text{if } P_{w_p}^{(n+2)} S_{w_1}^{(n+2)} = \emptyset \end{cases}$$

□

3.2 Skeleton and associated graphs

Let B be a finite rank Bratteli diagram with $|V_n| = d \geq 2$ for all levels n . We do not need to assume here that B is simple unless we state this explicitly. As usual, we denote by V the set of vertices of B but if one needs to point out that this set is considered at level n , then we write V_n instead of V . For $k \leq d$, take two subsets \tilde{V} and \overline{V} of V such that $|\tilde{V}| = |\overline{V}| = k$ (in particular, these two sets can coincide)².

²The assumption that $|\tilde{V}| = |\overline{V}|$ is made for convenience only; all definitions given below make sense for arbitrary sets \tilde{V} and \overline{V} .

Let $M_{\tilde{v}} = (M_{\tilde{v}}(1), \dots, M_{\tilde{v}}(n), \dots)$ and $m_{\bar{v}} = (m_{\bar{v}}(1), \dots, m_{\bar{v}}(n), \dots)$ be any two vertical paths in B going downward through the vertices $\tilde{v} \in \tilde{V}$ and $\bar{v} \in \bar{V}$. If $v \in \bar{V} \cap \tilde{V}$, then the paths M_v and m_v are taken such that they do not share common edges - recall that, without loss of generality, one can assume that every vertex in $\bigcup_{n=1}^{\infty} V_n$ is the range of at least two edges. Next, for each vertex $w \in V_n$, we choose two distinct edges \tilde{e}_w and \bar{e}_w such that $s(\tilde{e}_w) \in \tilde{V}_{n-1}$ and $s(\bar{e}_w) \in \bar{V}_{n-1}$. If $w \in \tilde{V}_n$ or $w \in \bar{V}_n$, then the edges \tilde{e}_w and \bar{e}_w are chosen such that $\tilde{e}_w = M_w(n)$ and $\bar{e}_w = m_w(n)$, respectively. Note that our choice of edges \tilde{e}_w and \bar{e}_w is not, in general, stationary and depends on the level n . We introduce the concepts of *skeleton* and *associated sequence of directed graphs* to create a framework for defining a perfect ordering with precisely these extremal paths.

Definition 3.6. Given a finite rank diagram B and two sets \tilde{V}, \bar{V} of the same cardinality, a *skeleton* $\mathcal{F} = \mathcal{F}(B)$ of B is a collection $\{M_{\tilde{v}}, m_{\bar{v}}, \tilde{e}_w, \bar{e}_w : w \in V \setminus \{v_0\}, \tilde{v} \in \tilde{V}, \text{ and } \bar{v} \in \bar{V}\}$ of paths and edges with the properties described above.

In other words, while not an ordering, a skeleton is a constrained choice of all extremal edges. As an example, when $\tilde{V} = \bar{V} = V$, the skeleton is simply the set $\{M_{\tilde{v}}, m_{\bar{v}} : \tilde{v}, \bar{v} \in V\}$. Our motivation to study skeletons is based on the following observation: *any ordered finite rank Bratteli diagram (B, ω) has a natural skeleton $\mathcal{F}_{\omega}(B)$.* To see this, we first notice that the extremal paths can be assumed to be vertical. Thus, any maximal path $M_v = (M_v(i))$ is defined by a vertex $v \in \tilde{V}$ and, similarly, a minimal path $m_{v'} = (m_{v'}(i))$ is determined by $v' \in \bar{V}$. The set $\{M_{\tilde{v}} = (M_{\tilde{v}}(i)) : \tilde{v} \in \tilde{V}\}$ of maximal paths and the set $\{m_{\bar{v}} = (m_{\bar{v}}(i)) : \bar{v} \in \bar{V}\}$ of minimal paths form a part of the skeleton $\mathcal{F}_{\omega}(B)$. The vertices from \tilde{V} will be called *maximal* and those from \bar{V} *minimal*.

We claim the following: for any level n there exist $l_n > n$ such that for every $l \geq l_n$ and every vertex $u \in V_l$ the maximal finite path $\tilde{e} = (\tilde{e}_1, \dots, \tilde{e}_l)$ and the minimal finite path $\bar{e} = (\bar{e}_1, \dots, \bar{e}_l)$ taken in the set $E(v_0, u)$ have the following property:

$$\tilde{e}_t = M_{\tilde{v}}(t), \quad \bar{e}_t = m_{\bar{v}}(t), \quad t = 1, \dots, n$$

where vertices $\tilde{v} \in \tilde{V}$ and $\bar{v} \in \bar{V}$ depend on u and l . Indeed, if we assumed that the contrary holds, then we would have additional maximal (or minimal) paths not belonging to $\{M_{\tilde{v}}\}$ (or $\{m_{\bar{v}}\}$) - see Proposition 3.4 for details. Moreover, the following condition holds (after an appropriate telescoping): if w is any vertex in V_n , $n \geq 2$, and \tilde{e}_w and \bar{e}_w are the maximal and minimal edges in the set $r^{-1}(w)$ with respect to ω , then $\tilde{e}_w \neq \bar{e}_w$ and $s(\tilde{e}_w) \in \tilde{V}_{n-1}$, $s(\bar{e}_w) \in \bar{V}_{n-1}$. Clearly, $\tilde{e}_{\tilde{v}} = M_{\tilde{v}}(n)$ if $\tilde{v} \in \tilde{V}_n$ and $\bar{e}_{\bar{v}} = m_{\bar{v}}(n)$ if $\bar{v} \in \bar{V}_n$. Thus, the ordering ω determines a collection $(M_{\tilde{v}}, m_{\bar{v}}, \tilde{e}_w, \bar{e}_w)$ and therefore \mathcal{F}_{ω} is completely defined.

Conversely, it is obvious that for any skeleton \mathcal{F} of a Bratteli diagram B there

is at least one ordering ω on B such that $\mathcal{F} = \mathcal{F}_\omega$. A skeleton \mathcal{F} contains no information about whether $\omega \in \mathcal{P}_B$, for $\mathcal{F} = \mathcal{F}_\omega$.

We now define a sequence of directed graphs $(\mathcal{H}_n = (T_n, P_n))$ associated to a Bratteli diagram B of finite rank and skeleton \mathcal{F} . Implicit in the definition of these directed graphs is the assumption that we are working towards constructing perfect orderings ω whose skeleton $\mathcal{F}_\omega = \mathcal{F}$. Thus we assume that $|\tilde{V}| = |\bar{V}|$ and we suppose that we also have a bijection $\sigma : \tilde{V} \rightarrow \bar{V}$ that, in the case when $\mathcal{F} = \mathcal{F}_\omega$ with $\omega \in \mathcal{P}_B$, will be the bijection described in Proposition 3.2. The graph $\mathcal{H}_n = (T_n, P_n)$ will be associated to the n -th level of B , and in general will vary as n changes; nevertheless we drop the index n unless it is explicitly needed.

In a general setting, let V be a finite set, \tilde{V} and \bar{V} two subsets of V of the same cardinality $k \leq |V|$, and $\sigma : \tilde{V} \rightarrow \bar{V}$ is a bijection. Let $W = \{W_{\tilde{v}} : \tilde{v} \in \tilde{V}\}$ and $W' = \{W'_{\bar{v}} : \bar{v} \in \bar{V}\}$ be two partitions of the set V . Define the set T of vertices of \mathcal{H} :

$$T = \{(\bar{v}, \tilde{v}) \in \bar{V} \times \tilde{V} : W'_{\bar{v}} \cap W_{\tilde{v}} \neq \emptyset\}.$$

To define the set P of directed edges of \mathcal{H} , we say that an arrow from (\bar{v}, \tilde{v}) to (\bar{v}_1, \tilde{v}_1) exists if $\sigma(\tilde{v}) = \bar{v}_1$. In general, the graph $\mathcal{H} = (T, P)$ is not connected.

We apply the definition of \mathcal{H} to a finite rank Bratteli diagram $B(V, E)$ with a given skeleton $\mathcal{F} = \{M_{\tilde{v}}, m_{\bar{v}}, \tilde{e}_w, \bar{e}_w : w \in V \setminus \{v_0\}, \tilde{v} \in \tilde{V}, \text{ and } \bar{v} \in \bar{V}\}$, where \tilde{V} and \bar{V} are of the same cardinality, and where we are also given a bijection $\sigma : \tilde{V} \rightarrow \bar{V}$. For any vertices $\tilde{v} \in \tilde{V}_{n-1}$ and $\bar{v} \in \bar{V}_{n-1}$, we set

$$W_{\tilde{v}}(n) = \{w \in V_n : s(\tilde{e}_w) = \tilde{v}\}, \quad W'_{\bar{v}}(n) = \{w \in V_n : s(\bar{e}_w) = \bar{v}\} \quad (3.4)$$

where $n \geq 2$. It is obvious that $W(n) = \{W_{\tilde{v}}(n) : \tilde{v} \in \tilde{V}_{n-1}\}$ and $W'(n) = \{W'_{\bar{v}}(n) : \bar{v} \in \bar{V}_{n-1}\}$ form two partitions of V_n . The intersection of $W(n)$ and $W'(n)$ is the partition $W'(n) \cap W(n)$ whose elements are non-empty sets $W'_{\bar{v}}(n) \cap W_{\tilde{v}}(n)$ where $(\bar{v}, \tilde{v}) \in \bar{V} \times \tilde{V}$. Given this information, we can now define the sequence of directed graphs $\mathcal{H}_n = \mathcal{H}_n(\mathcal{F}, \sigma)$ in the same way as the graph \mathcal{H} was defined. The sequence (\mathcal{H}_n) is called *the sequence of graphs associated to a finite rank B , \mathcal{F} , and σ* . It is worth noting that if B has a perfect ordering ω with the corresponding Vershik map φ_ω , then the skeleton \mathcal{F}_ω and map $\sigma : \tilde{V} \rightarrow \bar{V}$ are automatically defined such that $\varphi_\omega(M_{\tilde{v}}) = m_{\sigma(\tilde{v})}$.

Remark 3.7. Suppose (B, ω) is an ordered Bratteli diagram of finite rank and \mathcal{F}_ω is the skeleton on B defined by ω . Construct the sequence of associated graphs (\mathcal{H}) (we assume, without loss of generality, that $\mathcal{H} = (T, P)$ does not depend on levels). Then any vertex $v \in V$ belongs to a set $W'_{\bar{v}} \cap W_{\tilde{v}}$, i.e., to a vertex $t = (\bar{v}, \tilde{v}) \in T$ of the associated graph \mathcal{H} . Let $w = v_1 \cdots v_p$ be a word in the language $\mathcal{L}_{B, \omega}$; suppose $v_i \in t_i$ where $t_i \in T$. Then there exists a path in \mathcal{H} starting with t_1 and ending with

t_p . Clearly, this path may contain loops. Moreover, the following is also true; the proof is straightforward and is omitted.

Lemma 3.8. *Let B be an aperiodic finite rank Bratteli diagram, \mathcal{F} be a skeleton on B and $\sigma : \tilde{V} \rightarrow \overline{V}$ be a bijection. Let $(\mathcal{H}_n) = ((T_n, P_n))$ be the associated directed graphs. Suppose there exists an ordering ω on B with skeleton \mathcal{F} , such that for any $n, v \in V_n$ and $N > n$, if a word $w = v_1 \dots v_p \subset w(v, n, N)$, then w corresponds to a path in \mathcal{H}_n going through vertices t_1, \dots, t_p , where $v_i \in V_n$ belong to $t_i \in T_n$. Then ω is perfect and $\phi_\omega(M_{\tilde{v}}) = m_{\sigma(\tilde{v})}$.*

Definition 3.9. We define the family \mathcal{A} of Bratteli diagrams, all of whose incidence matrices are of the form

$$F_n := \begin{pmatrix} A_n^{(1)} & 0 & \dots & 0 & 0 \\ 0 & A_n^{(2)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n^{(k)} & 0 \\ B_n^{(1)} & B_n^{(2)} & \dots & B_n^{(k)} & C_n \end{pmatrix}$$

where

1. for $1 \leq i \leq k$ there is some d_i such that for each n , $A_n^{(i)}$ is a $d_i \times d_i$ matrix,
2. all matrices $A_n^{(i)}$, $B_n^{(i)}$ and C_n are strictly positive,
3. C_n is a $d \times d$ matrix,
4. there exists $j \in \{\sum_{i=1}^k d_i + 1, \dots, \sum_{i=1}^k d_i + d\}$ such that for each n , the j -th row of F_n is strictly positive.

As shown in [BKMS11], the family \mathcal{A} of diagrams corresponds to aperiodic homeomorphisms of a Cantor set that have exactly k minimal components with respect to the tail equivalence relation \mathcal{E} .

Recall that a directed graph is *strongly connected* if for any two vertices v, v' , there are paths from v to v' , and also from v' to v . If at least one of these paths exist, then G is *weakly connected*, or just *connected*.

Proposition 3.10. *Let B be a Bratteli diagram of finite rank. Suppose ω is a perfect ordering on B that defines the skeleton \mathcal{F}_ω and the bijection σ from \tilde{V} to \overline{V} is defined by the Vershik map φ_ω .*

1. *If B is simple, then the associated graph \mathcal{H}_n is strongly connected for any n .*
2. *If $B \in \mathcal{A}$, then the associated graph \mathcal{H}_n is weakly connected for any n .*

Proof. We prove (1) - the proof of (2) is similar, if we focus on $w(v, n-1, n)$ where v is the vertex which indexes the strictly positive row in F_n . In case of simple diagrams, we can assume that all entries of F_n are positive for each n . To simplify our notation we will omit the index n . We need to show that for any two vertices $t = (\bar{v}, \tilde{v})$ and $t' = (\bar{v}', \tilde{v}')$ from the vertex set T of \mathcal{H} there exists a path from t to t' .

Claim 1. Let $w(u, n, n+1) = v_1 \cdots v_k$ be a word where $v_i \in W'_{\bar{v}_i} \cap W_{\tilde{v}_i}$, $i = 1, \dots, k$. Then there is a path from (\bar{v}_1, \tilde{v}_1) to (\bar{v}_k, \tilde{v}_k) going through the vertices (\bar{v}_i, \tilde{v}_i) , $i = 1, \dots, k$.

To see this, it suffices to note that $\sigma(\tilde{v}_i) = \bar{v}_{i+1}$ because of the fact that v_{i+1} follows after v_i in the word w . Then Claim 1 is a simple consequence of this remark.

Now, let T^* be a subset of T constituted by vertices of the form $(\bar{v}, s(\tilde{e}_{\bar{v}}))$ where $\bar{v} \in \bar{V}$. It is obvious that there is an edge from (\bar{v}, \tilde{v}) to $(\sigma(\tilde{v}), s(\tilde{e}_{\sigma(\tilde{v})}))$ in \mathcal{H} . Based on this observation, we can prove the following statement.

Claim 2. (a) For any $t \in T$ there exists $t^* \in T^*$ such that there is an edge from t to t^* . (b) For any $t' \in T$ and $t^* \in T^*$, there is a path from t^* to t' .

To see (a), if $t = (\bar{v}, \tilde{v})$, take $t^* = (\sigma(\tilde{v}), s(\tilde{e}_{\sigma(\tilde{v})}))$. To see that (b) holds, we will use Claim 1. Let $t^* = (\bar{v}^*, \tilde{v}^*)$. Consider $r^{-1}(\bar{v}^*)$ and note that by simplicity of B there exists an edge $e \in E(v, \bar{v}^*)$ where $v \in W'_{\bar{v}'} \cap W_{\tilde{v}'}$. The perfect ordering ω defines the word $w = \bar{v}^* \cdots v$. It follows from Claim 1 that there is a path from t^* to t' .

To complete the proof of the proposition, we apply Claim 2. \square

Remark 3.11. It is not hard to see that the converse statement to Proposition 3.10 is not true. There are examples of non-simple diagrams of finite rank whose associated graphs are strongly connected.

Note also that the assumption that ω is perfect is crucial. Moreover, there are examples of *simple* finite rank Bratteli diagrams and skeletons none of whose associated graphs are strongly connected. Indeed, let B be a stationary diagram with $V = \{a, b, c\}$ with the skeleton $\mathcal{F} = \{M_a, M_b, m_a, m_b; \tilde{e}_c, \bar{e}_c\}$ where $s(\tilde{e}_c) = b, s(\bar{e}_c) = a$. Let $\sigma(a) = a, \sigma(b) = b$. Constructing the associated graph \mathcal{H} , we see that there is no path from (b, b) to (a, a) . It can be also shown that there is no perfect ordering ω such that $\mathcal{F} = \mathcal{F}_\omega$. This observation complements Proposition 3.10 by stressing the importance of the strong connectedness of \mathcal{H}_n for the existence of perfect orderings.

We illustrate the definitions of skeletons and associated graphs with several examples that will be also used later.

Example 3.12. 1. Let (B, ω) be an ordered Bratteli diagram of rank d with exactly d vertical maximal and d vertical minimal paths. Then the skeleton \mathcal{F}_ω is formed by pairs of vertical paths (M_i, m_i) going downward through the vertex $i \in \{1, \dots, d\}$, and the sequence of associated graphs \mathcal{H}_n is constant, $\mathcal{H}_n = \mathcal{H}$.

Let σ be a permutation of the set $(1, 2, \dots, d)$. The graph \mathcal{H} is represented as a disjoint union of connected subgraphs generated by cycles of σ . If ω is perfect, then by Proposition 3.10, σ is cyclic. In this case, $W'_i \cap W_i = \{i\}$, so vertices of \mathcal{H} are $\{(i, i) : 1 \leq i \leq d\}$, and there is an edge from $\{i\}$ to $\{j\}$ if and only if $j = \sigma(i)$. Thus, the structure of \mathcal{H} is represented by the cyclic permutation σ .

2. Let \mathcal{F} be a skeleton on a simple rank d diagram B such that $\tilde{V} = \overline{V} = \{1, \dots, d-1\}$. The sequence of strongly connected associated graphs \mathcal{H}_n that can be associated to \mathcal{F} is one of two kinds. Below we describe the structure of one of these graphs $\mathcal{H} = \mathcal{H}_n$.

- (a) Suppose $s(\tilde{e}_d) = s(\bar{e}_d) = j$ where $1 \leq j \leq d-1$ is a fixed vertex and \tilde{e}_d and \bar{e}_d are ω -maximal and ω -minimal edges. Then $W'_i \cap W_i = \{i\}$ for $1 \leq i \leq d-1$, $i \neq j$, and $W'_j \cap W_j = \{j, d\}$. In \mathcal{H} then the vertices consist of $\{(i, i)\}_{i=1}^{d-1}$. For \mathcal{H} to be strongly connected, σ must be a cyclic permutation of $(1, \dots, d-1)$, and in this case there is an edge from (i, i) to (i', i') if and only if $i' = \sigma(i)$.
- (b) Suppose $s(\tilde{e}_d) = j \neq s(\bar{e}_d) = i$ where $1 \leq i, j \leq d-1$; we can assume that $i < j$. Here $W'_l \cap W_l = \{l\}$ for $1 \leq l \leq d-1$ and $W'_i \cap W_j = \{d\}$, so that $T = \{(l, l) : 1 \leq l \leq d-1\} \cup \{(i, j)\}$. Here also, for \mathcal{H} to be strongly connected, σ must be a cyclic permutation of $(1, \dots, d-1)$, and the edges described in (2a) form a subset of P . In addition there is an edge from $(\sigma^{-1}(i), \sigma^{-1}(i))$ to (i, j) , and also an edge from (i, j) to $(\sigma(j), \sigma(j))$.

3. Let (B, ω) be a stationary ordered Bratteli diagram defined on four letters (a, b, c, d) by the substitution $a \rightarrow acbdba, b \rightarrow bdcdbacb, c \rightarrow acdcb, d \rightarrow bdacda$. It is clear that there are two pairs of maximal and minimal paths going through vertices a and b , $\omega \in \mathcal{P}_B$ and $\varphi_\omega(M_a) = m_a, \varphi_\omega(M_b) = m_b$. This means that σ is trivial, i.e., $\sigma(a) = a, \sigma(b) = b$. Noting that $s(\tilde{e}_c) = b, s(\tilde{e}_d) = a, s(\bar{e}_c) = a, s(\bar{e}_d) = b$, we have the completely determined skeleton \mathcal{F}_ω .

To construct the associated graph \mathcal{H} , we find $W_a = (a, d), W_b = (b, c), W'_a = (a, c), W'_b = (b, d)$. Hence the vertices T of \mathcal{H} are $W'_a \cap W_a = \{a\}, W'_a \cap W_b = \{c\}, W'_b \cap W_a = \{d\}, W'_b \cap W_b = \{b\}$. The associated graph \mathcal{H} is shown in Figure 2.

4. Let $V_n = V = \{v_1, v_2, v_1^*, v_2^*, w_1, w_2\}$ and $\tilde{V} = \overline{V} = \{v_1, v_1^*\}$; let $\sigma(v_1) = v_1$. Suppose that $W'_{v_1} = \{v_1, v_2, w_1\}, W_{v_1} = \{v_1, v_2, w_2\}, W'_{v_1^*} = \{v_1^*, v_2^*, w_2\}$ and

$W_{v_1^*} = \{v_1^*, v_2^*, w_1\}$. Then the associated graph \mathcal{H} is strongly connected. We remark that this can be the skeleton of an aperiodic diagram with two minimal components living through the vertices $\{v_1, v_2\}$ and $\{v_1^*, v_2^*\}$ respectively.

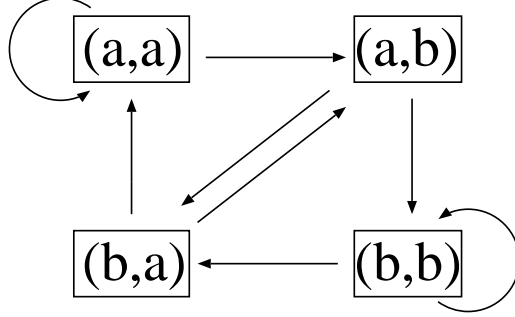


Figure 2: The graph associated to \mathcal{F}_ω in Example 3

We illustrate the utility of the notions of skeleton and accompanying directed graphs in the following results, which give sufficient conditions on an ordering ω so that it belongs to \mathcal{P}_B^c . Even though these are conditions on ω , some diagrams B force this condition on all orderings in \mathcal{O}_B - this is the content of Proposition 3.25.

Proposition 3.13. *Let the Bratteli diagram B have rank d , and be aperiodic. Suppose that ω is an ordering with k maximal and k minimal elements, where $1 < k < d$. Suppose also that for each vertex $v' \in V_n$ there is a vertex $v \in V_{n+1}$ such that $v'v' \in w(v, n, n+1)$. Then $\omega \notin \mathcal{P}_B$.*

Proof. We telescope B and ω to B' and $\omega' = L(\omega)$ so that (B', ω') has $\bar{V} = \{v_{m_1}, \dots, v_{m_k}\}$ and $\tilde{V} = \{v_{M_1}, \dots, v_{M_k}\}$ and skeleton $\{M_{\tilde{v}}, m_{\bar{v}}, \bar{e}_v, \tilde{e}_v\}$. We assume further that the telescoping is such that $s(\bar{e}_v)$ and $s(\tilde{e}_v)$ are independent of the level in which v lies. This means that $W(n) = W$, $W'(n) = W'$, and $\mathcal{H}_n = \mathcal{H}$.

Suppose that ω (and therefore ω') is perfect. Then there exists a bijection σ of $\{1, \dots, k\}$, such that $\sigma(i) = j$ if and only if $v_{M_i}v_{m_j} \in \mathcal{L}_{B', \omega'}$. We claim that $V = \bigcup_{i=1}^k (W_{v_{M_i}} \cap W'_{v_{m_{\sigma(i)}}})$. For if $s(\bar{e}_v) = v_{m_{\sigma(i)}}$ and $s(\tilde{e}_v) = v_{M_{i'}}$, for $i \neq i'$, then since for each n , there is a $v^* \in V_{n+1}$ with $vv^* \in w(v^*, n, n+1)$. This implies that $v_{M_{i'}}v_{m_{\sigma(i)}} \in w(v^*, n-1, n+1)$. The latter contradicts Proposition 3.2.

Since W and W' are partitions of V (i. e., $V = \coprod_{i=1}^k W_{v_{M_i}} = \coprod_{i=1}^k W'_{v_{m_{\sigma(i)}}}$), the relation $V = \bigcup_{i=1}^k (W_{v_{M_i}} \cap W'_{v_{m_{\sigma(i)}}})$ actually means that $W_{v_{M_i}} = W'_{v_{m_{\sigma(i)}}}$ for every i .

From the invariance of partition W with respect to σ , we conclude that the set $Y(i)$ of paths of B going through vertices of W_{M_i} is invariant with respect to the Vershik map φ_ω . Therefore $s(r^{-1}(v)) \in W_{v_{M_i}}$ for any $v \in W_{v_{M_i}}$. It follows that there exists j such that $V = W_{v_{M_j}} = W'_{v_{m_{\sigma(j)}}}$. This contradicts the assumption that $k > 1$. \square

It follows from the proof of Proposition 3.13 that the associated graph \mathcal{H} has the following simple form: the vertices of \mathcal{H} are $(v_{m_{\sigma(i)}}, v_{M_i}), i = 1, \dots, k$, and the edges are given by k loops around each vertex.

Proposition 3.13 is a special case of the following more general result.

Proposition 3.14. *Let the Bratteli diagram B have rank d . Suppose that $\omega \in \mathcal{P}_B$ is such that for the corresponding skeleton \mathcal{F}_ω we have*

1. $W(n) = W'(n)$ for each $n \in \mathbb{N}$, and
2. *There is a partition of the set $\{1, \dots, k\}$ into subsets, such that if $\{i_1, \dots, i_\ell\} \subset \{1, \dots, k\}$ is one such subset, then for each n , $W_{v_{M_{i_j}}}(n) = W'_{v_{m_{\sigma(i_{j-1})}}}(n)$ for $2 \leq j \leq \ell$, and $W_{v_{M_{i_1}}}(n) = W'_{v_{m_{\sigma(i_\ell)}}}(n)$.*

Then B is a disjoint union of some of its subdiagrams.

The *proof* of this result is based on the same idea as that of Proposition 3.13. Indeed, in Proposition 3.13, we have the equalities $W_{v_{M_j}} = W'_{v_{m_{\sigma(j)}}}$ and therefore deal with the partition of $\{1, \dots, k\}$ given by $\{\{1\}, \dots, \{k\}\}$. We note only that the more general situation the associated graph \mathcal{H}_n is now a disjoint union of circles: each set $\{i_1, \dots, i_\ell\}$ defines a subgraph as mentioned above whose vertices are linked by a cyclic permutation defined by σ .

We will now consider in details the class of finite rank diagrams described in Example 3.12 (1). Let an ordered Bratteli diagram B have rank $d > 1$. We show that if B is to support a perfect ordering with d maximal and d minimal paths, then a certain structure is imposed on the incidence matrices of B .

Denote by \mathcal{D} the set of rank d simple Bratteli diagrams whose incidence matrices (F_n) has the form:

$$F_n = \begin{pmatrix} f_1^{(n)} + 1 & f_1^{(n)} & \cdots & f_1^{(n)} \\ f_2^{(n)} & f_2^{(n)} + 1 & \cdots & f_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ f_d^{(n)} & f_d^{(n)} & \cdots & f_d^{(n)} + 1 \end{pmatrix} \quad (3.5)$$

where all entries are non-zero. It is not hard to check that the set \mathcal{D} is invariant under telescoping of diagrams.

Proposition 3.15. *Let $B \in \mathcal{D}$ be a simple Bratteli diagram of rank d with incidence matrices F_n .*

1. *Let σ be a cyclic permutation of the set $\{1, 2, \dots, d\}$. Then there exists a perfect ordering $\omega = \omega(\sigma)$ on B such that*

$$X_{\max}(\omega) = \{M_1, \dots, M_d\}, \quad X_{\min}(\omega) = \{m_1, \dots, m_d\}$$

where M_i (m_j) is a vertical paths going downward through the vertex v_i (v_j , respectively), $i, j = 1, \dots, d$. Moreover, the corresponding Vershik map φ_ω satisfies the condition

$$\varphi_\omega(M_i) = m_{\sigma(i)}. \quad (3.6)$$

2. Suppose there exists a perfect ordering ω on B such that $|X_{\max}(\omega)| = |X_{\min}(\omega)| = d$ and all maximal and minimal paths are vertical. Then the Vershik map φ_ω determines a cyclic permutation on the set $\{1, \dots, d\}$ and B belongs to \mathcal{D} , i.e., its incidence matrices satisfy (3.5) (possibly for all matrices below some level).

Proof. (1) We need to construct a perfect ordering ω on B such that (3.6) holds. For every $v_j \in \{v_1, \dots, v_d\} = V_n$ and $n \geq 2$, we take d subsets $E(v_i, v_j)$ of $r^{-1}(v_j)$ where $v_i \in V_{n-1}$. Then $|E(v_i, v_j)| = f_j^{(n)}$ if $i \neq j$ and $|E(v_j, v_j)| = f_j^{(n)} + 1$. Hence $|r^{-1}(v_j)| = df_j^{(n)} + 1$. For each n and each $v_j \in V_n$ define the order on $r^{-1}(v_j)$ as follows:

$$w(v_j, n-1, n) = (v_j v_{\sigma(j)} v_{\sigma^2(j)} \dots v_{\sigma^{d-1}(j)})^{f_j^{(n)}} v_j. \quad (3.7)$$

Clearly, relation (3.7) defines explicitly a linear order on $r^{-1}(v_j)$. To complete the definition of ω , we define the sets $X_{\max}(\omega), X_{\min}(\omega)$ formed by vertical paths M_i and m_j going through vertices of the diagram where $i, j = 1, \dots, d$. To see that φ_ω is continuous, it suffices to note that for each j there is a unique $i := \sigma(j)$ such that $v_j v_i \in \mathcal{L}_{B, \omega}$. By Part (1) of Proposition 3.2 we are done.

(2) Conversely, suppose that ω is a perfect ordering on B with d maximal and d minimal vertical elements, so that each vertex has to support both a maximal and a minimal path M_i and m_i ; thus for each i and each n , the word $\omega(v_i, n-1, n)$ starts and ends with v_i . Since ω is perfect then by Proposition 3.2 there is a permutation σ such that for each $j \in \{1, \dots, d\}$ only $v_j v_{\sigma(j)} \in \mathcal{L}_{B, \omega}$. So, for each j and all but finitely many n , there is a $f_j^{(n)}$ such that

$$w(v_j, n-1, n) = (v_j v_{\sigma(j)} v_{\sigma^2(j)} \dots v_{\sigma^{d-1}(j)})^{f_j^{(n)}} v_j. \quad (3.8)$$

Since B is simple, σ has to be cyclic so that all vertices occur in the right hand side of (3.8). From (3.8) it also follows that all but finitely many of the incidence matrices of B are of the form (3.5). \square

Corollary 3.16. *Let B be a simple Bratteli diagram of rank $d \geq 2$ and ω a perfect ordering on B with d maximal and minimal elements. Then (X_B, φ_ω) is conjugate to an odometer.*

Proof. Note that the proof of Theorem 3.15 tells us that $\mathcal{L}(B, \omega)$ is periodic. Lemma 3.5 tells us that (X_B, φ_ω) is conjugate to an odometer; however in this specific case

there is a simpler sequence of periodic, refining, generating partitions (Q_n) : let Q_n be the clopen partition defined by the first n levels of B . Since $Q_n = \coprod_{i=1}^d Q_n(v_i)$, where $Q_n(v_i)$ is the set of all paths from v_0 to $v_i \in V_n$, each non-maximal path in $Q_n(v_i)$ is mapped by φ_ω to its successor. For $i \in \{1, \dots, d\}$, let M_i denote the maximal path in $Q_n(v_i)$. Since the ordering ω is perfect, we obtain that $\varphi_\omega(M_i) = m_{\sigma(i)}$, where $m_{\sigma(i)}$ is the minimal path in $Q_n(v_{\sigma(i)})$. This means that the partition Q_n is φ_ω -periodic. We will also compute the sequence (k_n) such that $|Q_{n+1}| = k_n |Q_n|$. By Proposition 3.15, the incidence matrices of B are of the form (3.5): all columns of F_n sum to the same constant $k_n = (1 + \sum_{i=1}^d f_i^{(n)})$. Let $F_n = (f_{i,j}^{(n)})$ and $h_i^{(n)} := |Q_n(v_i)|$, then $h_i^{(n+1)} = \sum_{j=1}^d f_{i,j}^{(n)} h_j^{(n)}$ and

$$\begin{aligned} |Q_{n+1}| &= \sum_{i=1}^d h_i^{(n+1)} \\ &= \sum_{i=1}^d \left[h_i^{(n)} + \sum_{j=1}^d h_j^{(n)} f_i^{(n)} \right] \\ &= |Q_n| + \sum_{i=1}^d f_i^{(n)} \sum_{j=1}^d h_j^{(n)} \\ &= |Q_n| \left(1 + \sum_{i=1}^d f_i^{(n)} \right). \end{aligned}$$

□

We will discuss below the question under what condition a simple finite rank d Bratteli diagram B can have a perfect ordering ω with exactly $1 \leq k \leq d$ maximal (minimal) paths. It turns out that incidence matrices must satisfy certain conditions in this case.

Let (B, ω) be an ordered simple Bratteli diagram. Suppose ω is perfect and let $\varphi = \varphi_\omega$ be the corresponding Vershik map. Therefore $|\tilde{V}| = |\overline{V}|$ and φ_ω defines a one-to-one map $\sigma : \tilde{V} \rightarrow \overline{V}$ such that $\varphi_\omega(M_v) = m_{\sigma(v)}$, $v \in \tilde{V}$.

Fix $v \in \tilde{V}_{n-1}$ and $v' \in \overline{V}_{n-1}$ and consider two partitions $W = \{W_{\tilde{v}} : v \in \tilde{V}\}$ and $W' = \{W'_{\overline{v}} : v \in \overline{V}\}$ of V defined by ω as above.

We need some notation. Let $E(V_n, u)$ be the set of all finite paths between vertices of level n and a vertex $u \in V_m$ where $m > n$. The symbols $\tilde{e}(V_n, u)$ and $\bar{e}(V_n, u)$ are used to denote the maximal and minimal finite paths in $E(V_n, u)$, respectively. Fix maximal and minimal vertices \tilde{v} and \bar{v} . Denote $E(W_{\tilde{v}}(n), u) = \{e \in E(V_n, u) : s(e) \in W_{\tilde{v}}(n), r(e) = u\}$ and $\tilde{E}(W_{\tilde{v}}(n), u) = E(W_{\tilde{v}}(n), u) \setminus \{\tilde{e}(V_n, u)\}$. Similarly, $\overline{E}(W'_{\bar{v}}(n), u) = E(W'_{\bar{v}}(n), u) \setminus \{\bar{e}(V_n, u)\}$. Clearly, the sets $\{E(W_{\tilde{v}}(n), u) : \tilde{v} \in \tilde{V}\}$ form a partition of $E(V_n, u)$.

Lemma 3.17. *Let (B, ω) be an ordered finite rank simple diagram with a perfect ordering ω and let $\tilde{V}, \bar{V}, \sigma, W_{\tilde{v}}(n), W'_{\bar{v}}(n), \tilde{e}_w, \bar{e}_w$ be defined as above. Take a vertex $\tilde{v} \in \tilde{V}$. There exists n_0 such that for any $n \geq n_0$, any vertex $u \in V_m$ ($m > n$), and any finite path $e \in \tilde{E}(W_{\tilde{v}}(n), u)$ we have $\text{succ}(e) \in \bar{E}(W'_{\sigma(\tilde{v})}(n), u)$.*

Proof. Suppose that the conclusion of the lemma is not true. Then one can find a vertex $v \in \tilde{V}$, an infinite sequence $m_1 < m_2 < \dots$, vertices $u_n \in V_{k_n}$ ($k_n > m_n$), and finite paths $e_n \in \tilde{E}(W_v(m_n), u_n)$ such that $\text{Succ}(e_n) \notin \bar{E}(W'_{\sigma(v)}(m_n), u_n)$. On the other hand, there is $v'_n \in \bar{V}$, $v'_n \neq \sigma(v)$, such that $\text{Succ}(e_n) \in \bar{E}(W'_{v'_n}(m_n), u_n)$. Taking a subsequence of $\{m_n\}$ we can assume that for some $v' \in \bar{V}$ one has $v'_n = v'$ for all n . Then we obtain that the minimal path $m_{v'}$ belongs to $\text{Succ}(M_v)$. This contradicts to the existence of the Vershik map φ_ω since $v' \neq \sigma(v)$. \square

We immediately deduce from the lemma that the following result on entries of incidence matrices is true.

Corollary 3.18. *In the notation of Lemma 3.17, the following condition holds for the ordered simple diagram (B, ω) where $\omega \in \mathcal{P}_B$: for any $n \geq 2$, any vertex $\tilde{v} \in \tilde{V}_{n-1}$, any $m > n$, and any $u \in V_m$ one has*

$$|\tilde{E}(W_{\tilde{v}}(n), u)| = |\bar{E}(W'_{\sigma(\tilde{v})}(n), u)|.$$

As a matter of fact, this relation is true for all $n \geq n_0$. But we can telescope, if necessary, the diagram B contracting the first n_0 levels to satisfy this condition for all $n \geq 2$.

In particular, if B is as above, and (F_n) denotes the sequence of incidence matrices with non-zero entries $f_{v,w}^{(n)}$, then we can apply Corollary 3.18 to obtain the following property on F_n . Define two sequences of matrices $\tilde{F}_n = (\tilde{f}_{w,v}^{(n)})$ and $\bar{F}_n = (\bar{f}_{w,v}^{(n)})$ by the following rule (here $w \in V_{n+1}$, $v \in V_n$, $n \geq 1$):

$$\tilde{f}_{w,v}^{(n)} = \begin{cases} f_{w,v}^{(n)} - 1, & \tilde{e}_w \in E(v, w); \\ f_{w,v}^{(n)}, & \text{otherwise,} \end{cases} \quad (3.9)$$

$$\bar{f}_{w,v}^{(n)} = \begin{cases} f_{w,v}^{(n)} - 1, & \bar{e}_w \in E(v, w); \\ f_{w,v}^{(n)}, & \text{otherwise.} \end{cases} \quad (3.10)$$

Then for any $u \in V_{n+1}$, $\tilde{v} \in \tilde{V}_{n-1}$, we obtain that under conditions of Corollary 3.18 the entries of incidence matrices have the property:

$$\sum_{w \in W_{\tilde{v}}(n)} \tilde{f}_{u,w}^{(n)} = \sum_{w' \in W'_{\sigma(\tilde{v})}(n)} \bar{f}_{u,w'}^{(n)}, \quad n \geq 2. \quad (3.11)$$

The following result shows that condition (3.11) is sufficient to define a perfect ordering ω on a simple Bratteli diagram.

Theorem 3.19. *Let $B = (V, E)$ be a simple Bratteli diagram of finite rank d . Fix two subsets \tilde{V} and \overline{V} (each of cardinality k). Let a skeleton $\mathcal{F} = \{M_{\tilde{v}}, m_{\tilde{v}}, \tilde{e}_w, \bar{e}_w\}$ and a bijection $\sigma : \tilde{V} \rightarrow \overline{V}$ be such that all associated graphs \mathcal{H}_n are strongly connected. Suppose the entries of incidence matrices (F_n) satisfy condition (3.11) where the partitions $\{W_{\tilde{v}}(n) : \tilde{v} \in \tilde{V}\}$ and $\{W'_{\tilde{v}}(n) : \tilde{v} \in \tilde{V}\}$ are defined by the skeleton \mathcal{F} , and the entries $\tilde{f}_{w,v}^{(n)}, \bar{f}_{w,v}^{(n)}$ are defined according to (3.9), (3.10).*

Then there is a perfect ordering ω on B such that $\mathcal{F} = \mathcal{F}_\omega$ and the Vershik map φ_ω satisfies the relation $\varphi_\omega(M_{\tilde{v}}) = m_{\sigma(\tilde{v})}$.

Proof. Our goal is to define a linear order ω_u on $r^{-1}(u)$ for each $u \in V_{n+1}$ and $n > 1$ such that the corresponding partial ordering ω on B is perfect. Recall that each set $r^{-1}(u)$ contains two pre-selected edges \tilde{e}_u, \bar{e}_u and they should be the maximal and minimal edges after defining ω_u .

The proof is based on an inductive procedure that is applied to any row of incidence matrices. We first describe in details the first step of the algorithm that will be applied repeatedly. It will be seen from the proof of the theorem that for given \mathcal{F} and σ the order ω_u arising on $r^{-1}(u)$ is not unique.

We will first consider the particular case when the associated graphs $\mathcal{H} = (\mathcal{H}_n)$ defined by \mathcal{F} do not have loops. After that, we will modify the construction to include possible loops in the algorithm.

Case I: there is no loop in \mathcal{H} . To begin with, we take some $u \in V_{n+1}$ and consider the u -th rows of matrices \overline{F}_n and \tilde{F}_n . They coincide with $(f_{u,v_1}, \dots, f_{u,v_d})$ of matrix F_n except only one entry either corresponding to $|E(s(\bar{e}_u), u)|$ and $|E(s(\tilde{e}_u), u)|$. To simplify our notation, we omit the index n if this does not lead to a confusion. Take \bar{e}_u and assign the number 0 to it, i.e., \bar{e}_u is the minimal edge in ω_u . Let $(\tilde{v}_0, \tilde{v}_0)$ be the vertex³ of \mathcal{H} such that $s(\bar{e}_u) \in W'_{\tilde{v}_0} \cap W_{\tilde{v}_0}$. Consider the set $\{t \in \tilde{V} : (\sigma(\tilde{v}_0), t) \in \mathcal{H}\}$ (this set is formed by ranges of arrows in \mathcal{H} coming out from $(\tilde{v}_0, \tilde{v}_0)$). Find w' such that $f_{u,w'} \geq f_{u,w}$ for all entries $f_{u,w}, w \in W'_{\sigma(\tilde{v}_0)}$, (if there are several entries that are the maximal value, then $f_{u,w'}$ is chosen arbitrarily amongst them). Take any edge $e_1 \in E(w', u)$. In the case where $\tilde{e}_u \in E(w', u)$, we choose $e_1 \neq \tilde{e}_u$. Assign the number 1 to e_1 so that e_1 becomes the successor of $e_0 = \bar{e}_u$.

We note that in the collection of equations (3.11), numerated by vertices from \tilde{V} , we have worked with the equation defined by u and \tilde{v}_0 . Two edges were labeled in the above procedure, e_0 and e_1 . We may think of this step as if these edges were ‘removed’ from the set of all edges in $r^{-1}(u)$ so that the remaining non-enumerated

³The same word ‘vertex’ is used in two meanings: for elements of the set T of the graph \mathcal{H} and for elements of the set V of the Bratteli diagram B . To avoid a possible confusion, we point out explicitly what vertex is meant in the context.

edges satisfy the equation

$$\left(\sum_{w \in W_{\tilde{v}_0}} \tilde{f}_{u,w} \right) - 1 = \left(\sum_{w \in W'_{\sigma(\tilde{v}_0)}} \bar{f}_{u,w} \right) - 1. \quad (3.12)$$

We note also that the choice of w' from $W'_{\sigma(v_0)}$ actually means that we take some $\tilde{v}_1 \in \tilde{V}$ such that $s(e_1) \in W'_{\sigma(\tilde{v}_0)} \cap W_{\tilde{v}_1}$. In other words, we choose an arrow from (\bar{v}_0, \tilde{v}_0) to $(\sigma(\tilde{v}_0), \tilde{v}_1)$ in the associated graph \mathcal{H} . We claim that $\tilde{v}_1 \neq \tilde{v}_0$. For, if not, then $\sigma(\tilde{v}_1) = \sigma(\tilde{v}_0)$ but this implies that there would be a loop at $(\sigma(\tilde{v}_0), \tilde{v}_1)$, a contradiction to our assumption. Thus $\tilde{v}_1 \neq \tilde{v}_0$ and this is why there is exactly one edge removed from each side of (3.12), so that our resulting row still satisfies (3.11). This completes the first step of the construction.

We apply the described procedure again to show how we should proceed to complete the next step. Consider the set $\{f_{u,w} : w \in W'_{\sigma(\tilde{v}_1)}\}$ and find some w'' such that $f_{u,w''} \geq f_{u,w}$ for any $w \in W'_{\sigma(\tilde{v}_1)}$. In the corresponding set of edges $E(w'', u)$ we choose $e_2 \neq \tilde{e}_u$, and assign the number 2 to the edge e_2 , so that e_2 is the successor of e_1 .

Observe that now we are dealing with the equation of (3.11) that is determined by $\tilde{v}_1 \in \tilde{V}$. If we again ‘remove’ the enumerated edges e_1 and e_2 from it, then this equation remains true with both sides reduced by 1 as we saw the same in (3.12).

We remark also that the choice that we made of w'' (or e_2) allows us to continue the existing path (in fact, the arrow) in \mathcal{H} from (\bar{v}_0, \tilde{v}_0) to $(\sigma(\tilde{v}_0), \tilde{v}_1)$ and determine an arrow from $(\sigma(\tilde{v}_0), \tilde{v}_1)$ to $(\sigma(\tilde{v}_1), \tilde{v}_2)$ where \tilde{v}_2 is defined by the property that $s(e_2) \in W'_{\sigma(\tilde{v}_1)} \cap W_{\tilde{v}_2}$.

This process can be continued. At each step we apply the following rules:

- (1) the edge e_i , that must be chosen next after e_{i-1} , is taken from the set $E(w^*, u)$ where w^* corresponds to a maximal entry amongst $f_{u,w}$ where w runs over $W'_{\sigma(\tilde{v}_{i-1})}$;
- (2) the edge e_i is always taken not equal to \tilde{e}_u unless no more edges except \tilde{e}_u are left.

After every step of the construction, we see that the following statements hold.

(i) Equations (3.11) remain true when we treat them as the number of non-enumerated edges left in $r^{-1}(u)$. In other words, when a pair of vertices \tilde{v} and $\sigma(\tilde{v})$ is considered, we reduce by 1 each side of the equation defined by \tilde{v} .

(ii) The used procedure allows us to build a path p from the starting vertex (\bar{v}_0, \tilde{v}_0) going through other vertices of the graph \mathcal{H} according to the choice we make at each step. Since \mathcal{H} is strongly connected, the path will visit eventually every vertex of \mathcal{H} (in fact, many times depending on the entries of incidence matrix).

(iii) In accordance with (i), the u -th row of F_n is transformed by a sequence of steps in such a way that entries of the obtained rows form decreasing sequences. These entries show the number of non-enumerated edges remaining after the completed steps. It is clear that, by the rule used above, we decrease the largest entries

first. It follows from the simplicity of the diagram that, for sufficiently many steps, the set $\{s(e_i)\}$ will contain all vertices v_1, \dots, v_d from V_n . This means that the transformed u -th row consists of entries which are strictly less than those of F_n . After a number of steps the u -th row will have a form where the difference between any two entries is ± 1 . After that, this property will remain true.

(iv) It follows from (iii) that we finally obtain that all entries of the resulting u -th row are zeros or ones. We apply the same procedure to enumerate the remaining edges from $r^{-1}(u)$ such that the number $|r^{-1}(u)| - 1$ is assigned to the edge \tilde{e}_u . This means that we constructed the word $W_u = s(\tilde{e}_u)s(e_1) \cdots s(\tilde{e}_u)$ that determines an order on $r^{-1}(u)$.

Looking at the path p that is simultaneously built in \mathcal{H} , we see that the number of times this path comes into and leaves a vertex (\bar{v}, \tilde{v}) of the graph is in correspondence with (3.11). In other words, p is an Eulerian path of \mathcal{H} that finally arrives to the vertex of \mathcal{H} defined by $s(\tilde{e}_u)$.

Case II: there is a loop in \mathcal{H} . To deal with this case, we have to refine the described procedure to avoid a possible situation when the algorithm cannot be finished properly. Suppose that the graph \mathcal{H} has some loops. Without loss of generality, we may assume that there is exactly one loop around a vertex $t = (\bar{a}, \tilde{a}) \in T$ of \mathcal{H} . This means that $\sigma(\tilde{a}) = \bar{a}$. Let the corresponding set $W'_a \cap W_{\tilde{a}}$ consist of vertices b_1, \dots, b_q of the diagram. Again, we start with the first step as in (I) and construct a path p in \mathcal{H} that begins at the vertex (\bar{v}_0, \tilde{v}_0) of \mathcal{H} containing $s(\tilde{e}_u)$. As described above, we enumerate edges from $r^{-1}(u)$ and construct simultaneously the path p . Suppose that after j steps the path p arrives at (\bar{a}, \tilde{a}) for the first time. This means that $s(e_j) \in \{b_1, \dots, b_q\}$ but $s(e_i) \notin \{b_1, \dots, b_q\}$ for $0 \leq i < j$ (here we assume wlog that $(\bar{v}_0, \tilde{v}_0) \neq (\bar{a}, \tilde{a})$). At this moment, we need to change the procedure described in (I) and proceed as follows. Looking at the relation

$$\sum_{w \in W_{\tilde{a}}} \tilde{f}_{u,w} = \sum_{w' \in W'_{\sigma(\tilde{a})}} \bar{f}_{u,w'}, \quad (3.13)$$

we see that, firstly, $\tilde{f}_{u,w} = f_{u,w}$ and $\bar{f}_{u,w'} = f_{u,w'}$, and, secondly, the numbers f_{u,b_i} belong to both parts of (3.13) and can take arbitrary values because they cancel. Let $E_u(t) = E_u(b_1, \dots, b_q)$ be the subset of edges in $r^{-1}(u)$ with sources in the set $\{b_1, \dots, b_q\}$. Then $C = |E_u(t)| = \sum_{w=b_1}^{b_q} f_{u,w}$. Now, we consequently enumerate all edges from $E_u(b_1, \dots, b_q)$ in *arbitrary order* beginning with e_j . More precisely, we set $e_j < e_{j+1} < \cdots < e_{j+C-1}$ where the edges $e_{j+1}, \dots, e_{j+C-1} \in E_u(b_1, \dots, b_q)$, $e \neq e_j$, are taken in an arbitrary order. The corresponding path p will be a multiple loop about $t = (\bar{a}, \tilde{a})$. Then we return to the procedure from (I): when all edges from $E_u(b_1, \dots, b_q)$ are enumerated, the successor of e_{j+C-1} is taken from $W'_{\sigma(\tilde{a})}$ that contains the biggest entry. If we look at the row formed by the number of remaining

non-enumerated edges, we see that all (u, b_i) -entries of this row are zeros. Note that the graph \mathcal{H} remains strongly connected after excluding a loop. Indeed suppose that there is an edge from $t = (\bar{a}, \tilde{a})$ to (\bar{b}, \tilde{b}) , and also an edge from (\bar{c}, \tilde{c}) to (\bar{a}, \tilde{a}) . Then $\sigma(\tilde{a}) = \bar{b}$ but also $\sigma(\tilde{a}) = \bar{a}$, and $\sigma(\tilde{c}) = \bar{a} = \bar{b}$, so there is an edge from (\bar{c}, \tilde{c}) to (\bar{b}, \tilde{b}) .

Clearly, we can apply the above construction to every loop independently, so that the assumption that the loop is unique is not crucial.

To summarize Cases I and II, we notice that, constructing the Eulerian path p , the following rule is used: as soon as p arrives at a loop around a vertex t in \mathcal{H} , then p makes as many loops around t as the cardinality of the set of edges $E_u(t)$. Then p leaves t and goes to the vertex t' according to the procedure in Case I.

As noticed above, the fact that all edges e from $r^{-1}(u)$ are enumerated is equivalent to defining a word formed by the sources of e . In our construction, we obtain the word $w(u, n, n+1) = s(\bar{e}_u)s(e_1) \cdots s(e_j) \cdots s(e_{j+C}) \cdots s(\tilde{e}_u)$.

Applying these arguments to every vertex u of the diagram, we define an ordering ω on B . That ω is perfect follows from Lemma 3.8: we chose ω to have skeleton \mathcal{F} , and for each n , constructed all words $w(v, n, n+1)$ to correspond to paths in \mathcal{H}_n . The result follows. \square

Remark 3.20. We observe that the assumption about simplicity of the Bratteli diagram in the above theorem is redundant. It was used only when we worked with strictly positive rows of incidence matrices. But for a non-simple finite rank diagram B we can use the following result proved in [BKMS11].

Any Bratteli diagram of finite rank is isomorphic to a diagram whose incidence matrices $(F_n)_n$ are of the form

$$F_n = \begin{pmatrix} F_1^{(n)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & F_2^{(n)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_s^{(n)} & 0 & \cdots & 0 \\ X_{s+1,1}^{(n)} & X_{s+1,2}^{(n)} & \cdots & X_{s+1,s}^{(n)} & F_{s+1}^{(n)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ X_{m,1}^{(n)} & X_{m,2}^{(n)} & \cdots & X_{m,s}^{(n)} & X_{m,s+1}^{(n)} & \cdots & F_m^{(n)} \end{pmatrix}. \quad (3.14)$$

For every $n \geq 1$, the matrices $F_i^{(n)}$, $i = 1, \dots, s$, have strictly positive entries and matrices $F_i^{(n)}$, $i = s+1, \dots, m$, have either strictly positive or zero entries. For every fixed $j = s+1, \dots, m$, there is at least one non-zero matrix $X_{j,k}^{(n)}$.

It follows from (3.14) that, for $u \in V_{n+1}$, the u -th row of F_n consists of several parts such that the proof of Theorem 3.19 can be applied to each of these parts independently. Indeed, this is obvious that if u belongs to any subdiagram defined

by $(F_i^{(n)})$, $i = 1, \dots, s$, then we have a simple subdiagram. If u is taken from $(F_i^{(n)})$, $i = s + 1, \dots, m$, then by (3.14) we may have some zeros in a row but they do not affect the procedure in the proof of Theorem 3.19.

We illustrate the proof of Theorem 3.19 with the following examples.

Example 3.21. Suppose B is a rank 6 Bratteli diagram defined on the vertices $\{a, b, c, d, e, g\}$. Let $\bar{V} = \tilde{V} = \{a, b, c\}$ and $\sigma(a) = b, \sigma(b) = c, \sigma(c) = a$. Take the skeleton $\mathcal{F} = \{M_a, M_b, M_c, m_a, m_b, m_c; \bar{e}_d, \tilde{e}_d, \bar{e}_e, \tilde{e}_e, \bar{e}_g, \tilde{e}_g\}$ where $s(\bar{e}_d) = b$, $s(\bar{e}_e) = b$, $s(\bar{e}_g) = c$ and $s(\tilde{e}_d) = a$, $s(\tilde{e}_e) = a$, $s(\tilde{e}_g) = c$. For simplicity of notation, we suppose that \mathcal{F} and σ are stationary. For such a choice of the data, we see that non-empty intersections of partitions W and W' give the following sets: $W'_a \cap W_a = \{a\}$, $W'_b \cap W_a = \{d, e\}$, $W'_b \cap W_b = \{b\}$, $W'_c \cap W_c = \{c, g\}$. The graph \mathcal{H} is illustrated in Figure 3.

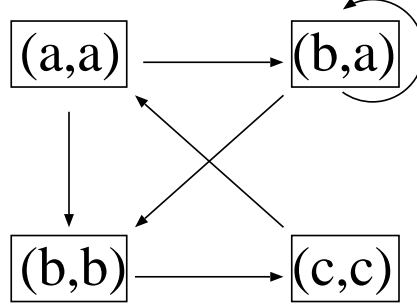


Figure 3: The graph associated to \mathcal{F}_ω in Example 3.21

We see that \mathcal{H} has four vertices and one loop around the vertex (b, a) . The directed edges are shown on the figure and defined by σ .

We consider, for definiteness, the case $u = a$ only and construct an order on $r^{-1}(a)$ according to Theorem 3.19. In this case, conditions (3.11) have the form: $f_{a,a} - 1 = f_{a,b} = f_{a,c} + f_{a,g}$ and the entries $f_{a,d}, f_{a,e}$ can be taken arbitrary because they correspond to the loop in \mathcal{H} . For instance, the following row $(3, 2, 1, 3, 2, 1)$ satisfies the above condition. Applying the algorithm offered in the proof of the theorem, we can order the edges from $r^{-1}(a)$ such that their sources form the word $w(a, n-1, n) = addeedbgabca$. To define an order on $r^{-1}(v)$, $v = b, c, d, e, g$, we apply similar arguments (details are left to the reader). By Theorem 3.19, we conclude that if the entries of incidence matrices satisfy (3.11), then B admits a perfect ordering ω such that $\mathcal{F} = \mathcal{F}_\omega$ and the Vershik map agrees with σ .

In the next example, we will show how one can describe the structure of Bratteli diagrams of rank d for which there exists a perfect ordering with exactly $d - 1$ maximal and minimal paths. The following example deals with a finite rank 3 diagram.

Example 3.22. Suppose B is a rank 3 diagram defined on the vertices $\{a, b, c\}$ with $\overline{V} = \tilde{V} = \{a, b\}$ and $\sigma(a) = b, \sigma(b) = a$. Take the skeleton $\mathcal{F} = \{M_a, M_b, m_a, m_b; \bar{e}_d, \tilde{e}_c, \bar{e}_c\}$ where $s(\bar{e}_c) = b, s(\tilde{e}_c) = a$. We suppose, for simplicity, that \mathcal{F} and σ are stationary. For such a choice of the data, we see that $W'_a \cap W_a = \{a\}, W'_a \cap W_b = \emptyset, W'_b \cap W_a = \{c\}, W'_b \cap W_b = \{b\}$ and \mathcal{H} is illustrated in Figure 3.22.

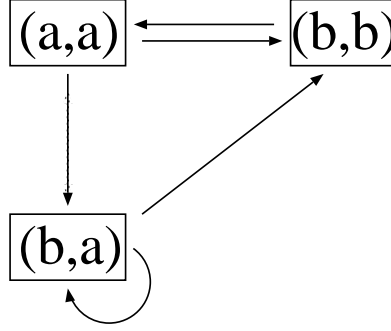


Figure 4: The graph associated to \mathcal{F} in Example 3.22

To satisfy the condition of Theorem 3.19, we have to take the incidence matrix

$$F = \begin{pmatrix} f+1 & f & p \\ g & g+1 & q \\ t & t & s \end{pmatrix}$$

where the entries are any positive integers. We note that the form of F depends on the given skeleton. In order to see how Theorem 3.19 works, one can choose some specific values for the entries of F and repeat the proof of the theorem. For example, if the incidence matrix is of the form

$$F = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 4 & 2 \end{pmatrix},$$

then one possibility for a valid ordering is $w(a, n-1, n) = abacba$, $w(b, n-1, n) = bacbab$ and $w(c, n-1, n) = baccbababa$. Note that when defining $w(b, n-1, n)$, we make a random choice for the third letter, and a random choice for the fifth letter of $w(c, n-1, n)$. We are constrained when choosing the fourth and eighth letters of $w(a, n-1, n)$ and $w(c, n-1, n)$ respectively.

Next we consider conditions for a Bratteli diagram B to support a perfect ordering which generates an odometer. Suppose for the time being that we are given a stationary skeleton \mathcal{F} : we have a rank d , sets \tilde{V} and \overline{V} both of cardinality $k \leq d$, a bijection $\sigma : \tilde{V} \rightarrow \overline{V}$, and partitions $W' = \{W'_{\bar{v}} : \bar{v} \in \overline{V}\}$ and $W = \{W_{\tilde{v}} : \tilde{v} \in \tilde{V}\}$. (Note that we have not yet specified a diagram B .) Let $\mathcal{H} = (T, P)$ be the directed

graph associated to \mathcal{F} . Let us assume that \mathcal{H} is strongly connected. Let W be a path in \mathcal{H} . Then W can correspond to several words in $V^+ = \{v_1, \dots, v_d\}^+$: for example if W starts at vertex (\tilde{v}, \tilde{v}) , then it corresponds to words starting with v whenever $v \in W_{\tilde{v}} \cap W_{\tilde{v}}'$. If W is a word in V^+ then we write W_v to mean that W ends with v , and ${}_v W$ to mean that W starts with v . If $W \in V^+$ corresponds to a path in \mathcal{H} , we write $W^{\mathcal{H}}$ to denote the corresponding path. It is not difficult to find words $W \in V^+$ corresponding to a path in \mathcal{H} such that

1. W contains all v_i 's,
2. W^2 corresponds to a legitimate path in \mathcal{H} , and
3. for each $\tilde{v} \in \tilde{V}$, there exist words $P_{\tilde{v}}$ and ${}_{\sigma(\tilde{v})}S$ such that $W = P_{\tilde{v}} {}_{\sigma(\tilde{v})}S$.

Call a word which satisfies 1-3 σ -decomposable. If W is a word, let \vec{W} be the d -dimensional vector whose i -th entry is the number of occurrences of $v_i \in V$.

Note that the requirement that the skeleton be stationary can be relaxed: it suffices to find a σ -decomposable word W that works for the sequence of directed graphs \mathcal{H} .

The following result generalizes Proposition 3.15, and gives the constraints on the sequence (F_n) of transition matrices that a diagram B has in order for B to support an odometer with a periodic language.

Proposition 3.23. *Suppose \mathcal{F} is such that \mathcal{H} is strongly connected, and let W be σ -decomposable. Let $\{p_v^{(n)}\}_{v \in V, n \in \mathbb{N}}$ be a set of nonnegative integers. Then there exist ω -ordered Bratteli diagrams B with skeleton \mathcal{F} , whose incidence matrices F_n have*

$$\vec{S}_{\tilde{v}} + p_v^{(n)} \vec{W} + \vec{P}_{\tilde{v}} \quad (3.15)$$

as the v -th row, whenever $v \in W_{\tilde{v}}' \cap W_{\tilde{v}}$; and such that (X_B, φ_ω) is topologically conjugate to an odometer.

Proof. Define, for $v \in W_{\tilde{v}}' \cap W_{\tilde{v}}$, $w(v, n-1, n) := \vec{S}_{\tilde{v}} W^{p_v^{(n)}} P_{\tilde{v}}$. Note that the v -th row of $F^{(n)}$ satisfies 3.15, and B has as (stationary) skeleton \mathcal{F} . Now \mathcal{H} tells us what words of length 2 are allowed in $\mathcal{L}_{B, \omega}$: $vv' \in \mathcal{L}_{B, \omega}$ if and only if $v \in W_{\tilde{v}}' \cap W_{\tilde{v}}$, $v' \in W_{\tilde{v}'}' \cap W_{\tilde{v}'}$, and $\sigma(\tilde{v}) = \tilde{v}'$. Thus

$$w(v, n-1, n)w(v', n-1, n) = \vec{S}_{\tilde{v}} W^{p_v^{(n)}} P_{\tilde{v}} \vec{S}_{\tilde{v}'} W^{p_{v'}^{(n)}} P_{\tilde{v}'} = \vec{S}_{\tilde{v}} W^{p_v^{(n)}} W W^{p_{v'}^{(n)}} P_{\tilde{v}'}$$

by Property 3 of a σ -decomposable word. Since $w(v, n-1, n+1)$ (and more generally, $w(v, n-1, N)$) is a concatenation of words $w(v, n-1, n)$, this implies that $\mathcal{L}_{B, \omega}$ is periodic. Proposition 3.5 implies the desired result. \square

There is a converse to this result: namely that if a perfect order ω on a simple diagram B has a periodic language, then there is some σ -decomposable word which generates $\mathcal{L}(B, \omega)$, so that by Lemma 3.5, (X_B, ϕ_ω) is an odometer.

If $V = \{v_1, v_2, \dots, v_d\}$ and a perfect ω is to have d maximal paths, then Proposition 3.15 tells us that $v_1 v_2 \dots v_d$ is, upto rotation, the only σ -decomposable word. The next example shows that in general σ -decomposable words are easy to find.

Example 3.24. Let $V = \{a_1, a_2, \dots, a_{n+1}\}$, $\overline{V} = \tilde{V} = \{a_1, a_2, \dots, a_n\}$, $\sigma(a_i) = a_{i+1}$ for $i < n$ and $\sigma(a_n) = a_1$, where $W'_{a_i} \cap W_{a_i} = \{a_i\}$ for each i and $a_{n+1} \in W_{a_j} \cap W'_{a_i}$ for some $j \neq i$. Then any word starting with a_i (for $1 \leq i \leq n$), ending with $\sigma^{-1}(a_i)$, and containing all a_i 's is σ -decomposable.

The next proposition describes how for some aperiodic diagrams B that belong to the special class \mathcal{A} (see Definition 3.9), there are structural obstacles to the existence of perfect orders on B . This is a generalization of an example in [Me06].

Proposition 3.25. *Let $B \in \mathcal{A}$ have k minimal components, and such that for each n , C_n is an $s \times s$ matrix where $1 \leq s \leq k - 1$. If $k = 2$, there are perfect orderings on B only if $C_n = (1)$ for all but finitely many n . If $k > 2$, then there is no perfect ordering on B .*

Proof. We use the notation of Definition 3.9 in this proof. Let V^i be the subset of vertices corresponding to the subdiagram defined by the matrices $A_n^{(i)}$ for $i = 1, \dots, k$, and V^{k+1} be the subset of vertices corresponding to the subdiagram defined by the matrices C_n . Note that if B has incidence matrices of the given form, then so does any telescoping. Suppose that ω is a perfect ordering, and we have a telescoping of B so that all extremal paths are vertical; let \mathcal{F}_ω be the skeleton for this telescoped diagram and ordering. Note that $|\overline{V}| = |\tilde{V}| \geq k$ since each minimal component has at least one maximal and one minimal path. For ease of notation we assume that \mathcal{F}_ω is stationary, so that $\mathcal{H}_n = \mathcal{H}$ for each n . There are k connected components of vertices T_1, \dots, T_k , such that there are no edges from vertices in T_i to vertices in T_j if $i \neq j$. To see this, if $1 \leq i \leq k$, let $T_i = \{(\overline{v}, \tilde{v}) : \overline{v} \in V^i, \tilde{v} \in V^i\}$.

If $k = 2$, there are no extremal paths going through c , the unique vertex in V^3 - otherwise there are disjoint components in \mathcal{H} , and this contradicts the fact that all $B_n^{(i)}$'s are positive. So $c \in W'_{\overline{v}} \cap W_{\tilde{v}}$ where $\overline{v} \in V^i$ and $\tilde{v} \in V^j$ for some $i \neq j$. Thus in H there are paths from vertices in T_i to vertices in T_j through c , but not back again. The only way this can occur validly is if $C_n = (1)$ for all n .

If $k > 2$, then there are at most $k - 1$ vertices remaining in \mathcal{H} , and for each one of these remaining vertices there are incoming edges from vertices in exactly one of the components T_i , for $1 \leq i \leq k$, and also outgoing edges to vertices in exactly one of the components T_i , for $1 \leq i \leq k$. So at least one of the components, say T_1 has no outgoing edges with range outside T_1 . But then if for $t = (\overline{v}, \tilde{v}) \in \mathcal{H}$, there are

no outgoing edges from t into vertices in T_1 , then for $v \in W_{\bar{v}}' \cap W_{\tilde{v}}$, $w(v, n, n+1)$ contains no occurrences of vertices from V^1 , contradicting the fact that the matrices $B_n^{(1)}$ are strictly positive. Also, if for each of the (at most) $k-1$ remaining vertices $t = (\bar{v}, \tilde{v})$, there are outgoing edges (only) to T_1 , this also contradicts the positivity of the matrices $B_n^{(1)}$. \square

In the above proposition, the extreme case - when there are k extremal pairs, and $|\mathcal{H}| = 2k-1$ - still does not produce perfect orderings, but only just, as the next proposition demonstrates. We abuse notation and use \mathcal{D} (defined in (3.5)) to denote not just a family of diagrams, but also the family of matrices of the form (3.5).

Proposition 3.26. *Let $B \in \mathcal{A}$ be a Bratteli diagram with k minimal subcomponents, and where for each n , C_n is a $k \times k$ matrix. If ω is a perfect ordering on B with skeleton \mathcal{F} , then $C_n \in \mathcal{D}$ for all n .*

Proof. We use the notation of Proposition 3.25: if \mathcal{H} is the directed graph associated to \mathcal{F} , then there are k connected components $T_1 \dots T_k$ of vertices where $(\bar{v}, \tilde{v}) \in T_i$ if and only if both \bar{v} and \tilde{v} belong to V^i . For the matrices $B_n^{(i)}$ to be strictly positive there must exist paths in \mathcal{H} from T_i to T_j for $1 \leq i \neq j \leq k$. This means that each other vertex (\bar{v}, \tilde{v}) corresponds to a single vertex in V^{k+1} (so that we label these vertices in \mathcal{H} using these vertices in V), that there exist bijections $h, h' : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ such that for each vertex $v_i \in V^{k+1}$ ($1 \leq i \leq k$) we have $v_i \in W_{\bar{v}_{h'(i)}}' \cap W_{\tilde{v}_{h(i)}}$ with $\bar{v}_{h'(i)} \in V^{h'(i)}$ and $\tilde{v}_{h(i)} \in V^{h(i)}$. Note that from each component T_i there is a unique outgoing edge (from T_i to $v_{h^{-1}(i)}$), a unique incoming edge (from $v_{h^{-1}(i)}$ to T_i). Also, for each $1 \leq i \leq k$, other than the incoming/outgoing edges from/to $T_{h'(i)}/T_{h(i)}$, there is an incoming edge from $v_{h^{-1}h'(i)}$ and an outgoing edge to $v_{h'^{-1}h(i)}$.

Thus for any $v_i \in V^{k+1}$ and any n , if $v_i \in W_{\bar{v}_{h'(i)}}' \cap W_{\tilde{v}_{h(i)}}$ then

$$w(v_i, n-1, n) = W(V^{h'(i)})v_i W(V^{h(i)})v_{i_1} W(V^{h(i_1)})v_{i_2} \dots v_{i_k} W(V^{h(i)})$$

where W^j is a word with letters in V^j , $i_1 = h'^{-1}h(i)$, $i_j = h'^{-1}h(i_j - 1)$ for $1 < j \leq k-1$ and $i_k = i$. The result follows. \square

4 The measurable space of orderings on a diagram

Recall that μ has been defined as the product measure on the set \mathcal{O}_B .

Theorem 4.1. *Let B be a finite rank d aperiodic Bratteli diagram. Then there exists $j \in \{1, \dots, d\}$ such that μ -almost all orderings have j maximal and j minimal elements.*

Proof. We shall first show that there exist j and j' such that μ -almost all orderings have j maximal and j' minimal elements. We then show that $j = j'$ in Corollary 4.3. If B has rank d , then for $k \in \mathbb{N}$, $1 \leq i \leq d$ and $n > k$, define the event

$$G_k^{n,i} = \{\omega : \text{the maximal paths from level } k \text{ to level } n \text{ have exactly } i \text{ distinct sources}\},$$

and

$$H_k^i := \bigcup_{n>k} G_k^{n,i}.$$

If

$$\mathcal{O}_B(j) := \{\omega : \omega \text{ has } j \text{ maximal paths}\},$$

then we claim that $\mathcal{O}_B(1) = \limsup H_k^1$. For if $\omega \in \limsup H_k^1$, then for some subsequence (n_k) , $\omega \in H_{n_k}^1 = \bigcup_{n>n_k} G_{n_k}^{n,1}$ for each k . For each n_k , there is some $n > n_k$ such that the maximal paths from level n_k to level n have only one source. This means there is only one maximal path from level 1 to level n_k that is extended to an infinite maximal path. Letting $n_k \rightarrow \infty$, we have that $\omega \in \mathcal{O}_B(1)$. Conversely, suppose that $\omega \notin \limsup H_k^1$. Then for some K , and all $k > K$,

$$\omega \in \left(\bigcup_{n>k} G_k^{n,1}\right)^c = \bigcap_{n>k} \bigcup_{i=2}^d G_k^{n,i}.$$

Fix $k > K$. For some j , and some $\{v_1 \dots v_j\} \subset V_k$, we have $\omega \in G_k^{n_p,j}$ for infinitely many $n_p > k$, where the sources of the maximal paths from level k to level n_p are $\{v_1 \dots v_j\}$ for each of these n_p 's. Fix n_1 ; for some set $\{v_1^1, \dots, v_j^1\} \subset V_{n_1}$, and for some subsequence $(n_{p(1)})$ of (n_p) , there are j maximal paths from level k to level $n_{p(1)}$ whose sources are $\{v_1 \dots v_j\}$ and which pass through $\{v_1^1, \dots, v_j^1\} \subset V_{n_1}$, for any $n_{p(1)}$. Let $\{M_1^{(i)} : 1 \leq i \leq d\}$ be the maximal paths from level k to level n_1 with $r(M_1^{(i)}) = v_i^1$ for $1 \leq i \leq j$. Fix one n_2 in $(n_{p(1)})$. There exists $\{v_1^2, \dots, v_j^2\} \subset V_{n_2}$ and $(n_{p(2)})$, a subsequence of $(n_{p(1)})$, such that for each $n_{p(2)}$, there are j maximal paths from level k to level $n_{p(2)}$ with range $\{v_1^2, \dots, v_j^2\} \subset V_{n_2}$. Let $\{M_2^{(i)} : 1 \leq i \leq d\}$ be these maximal paths. Each $M_2^{(i)}$ is a refinement of $M_1^{(i)}$. Continue in this fashion to get for each $1 \leq i \leq j$ a sequence $(M_j^{(i)})$ of paths converging to j distinct maximal paths, so that $\omega \notin \mathcal{O}_B(1)$.

Similarly we can show that for $1 < j \leq d$,

$$\mathcal{O}_B(j) = \left(\limsup_{k \rightarrow \infty} H_k^j\right) \setminus \bigcup_{i=1}^{j-1} \mathcal{O}_B(i).$$

Now order the vertices in $V = \bigcup_{n \geq 2} V_n$ as $\{v_1, v_2, \dots\}$ starting from level 2 and moving to levels V_n , $n = 3, 4, \dots$. For each n define the random variable X_n on \mathcal{O}_B where $X_n(w) = i$ if the source of the maximal edge with range v_n is the vertex

i. The sequence (X_n) is a sequence of mutually independent variables and if Σ_n is the σ -field generated by $\{X_n, X_{n+1}, \dots\}$ and $\Sigma := \bigcap_n \Sigma_n$, then for each $1 \leq i \leq d$, $\mathcal{O}_B(j) \in \Sigma$ and by Kolmogorov's zero-one law, for each $1 \leq j \leq d$, $\mu(\mathcal{O}_B(j))$ is either 0 or 1. Note now that one can repeat the definitions of all the above sets replacing the word 'maximal' with 'minimal'. The result follows. \square

In the next result we use our notation from the proof of Theorem 4.1.

Theorem 4.2. *Let B be an aperiodic Bratteli diagram of rank d .*

1. $\mu(\mathcal{O}_B(1)) = 1$ if and only if there exists a sequence $(n_k)_{k=1}^\infty$ such that $\sum_{k=1}^\infty \mu(G_{n_k}^{n_{k+1},1}) = \infty$.
2. Let $1 < j \leq d$. Then $\mu(\mathcal{O}_B(j)) = 1$ if and only if there exists a sequence (n_k) where $\sum_k \mu(G_{n_k}^{n_{k+1},j}) = \infty$, and for each $1 \leq i < j$, and all sequences (m_k) , $\sum_k \mu(G_{m_k}^{m_{k+1},i}) < \infty$.

Proof. (1) Note that for each j and n with $n > j$,

$$G_j^{n,1} \subset G_j^{n+1,1} \quad (4.16)$$

and similarly for each j , n with $n > j+1$, $G_{j+1}^{n,1} \subset G_j^{n,1}$. This implies that

$$H_{j+1}^1 = \bigcup_{n>j+1} G_{j+1}^{n,1} \subset \bigcup_{n>j+1} G_j^{n,1} \subset \bigcup_{n>j} G_j^{n,1} = H_j^1. \quad (4.17)$$

If $\mu(\mathcal{O}_B(1)) = 1$, then since from the proof of Theorem 4.1 $\mathcal{O}_B(1) = \limsup H_k^1$, we have

$$1 = \mu(\mathcal{O}_B(1)) = \mu\left(\bigcap_{k=1}^\infty \bigcup_{j \geq k} H_j^1\right) \stackrel{(4.17)}{=} \mu\left(\bigcap_{k=1}^\infty H_k^1\right),$$

which implies that for each k , $\mu(H_k^1) = 1$, and now Inclusion (4.16) implies that for each k ,

$$1 = \mu(H_k^1) = \mu\left(\bigcup_{n>k} G_k^{n,1}\right) = \lim_{n \rightarrow \infty} \mu(G_k^{n,1}), \quad (4.18)$$

and this implies the existence of a sequence (n_k) such that $\sum_{k=0}^\infty \mu(G_{n_k}^{n_{k+1},1}) = \infty$.

Conversely, suppose there is some (n_k) such that $\sum_k \mu(G_{n_k}^{n_{k+1},1}) = \infty$. The converse of the Borel-Cantelli lemma implies that for μ -almost all orderings, there is a subsequence (j_k) such that all maximal edges in E_{j_k} have the same source. This implies that for almost all ω there is at most one, and thus exactly one maximal path in X_B .

(2) We prove Statement (2) for $j = 2$, other cases follow similarly. If $\mu(\mathcal{O}_B(2)) = 1$, then $\mu(\mathcal{O}_B(1)) = 0$, and by the proof of Theorem 4.1, this means that $\mu(\limsup H_k^2) = 1$ and $\mu(\limsup H_k^1) = 0$. Using (1), we conclude that for all

sequences (m_k) , $\sum_k \mu(G_{m_k}^{m_{k+1},1}) < \infty$. Also, as in the proof of (1), we will have that for each k ,

$$\lim_{n \rightarrow \infty} \mu(G_k^{n,1}) = 0.$$

Note that for all $n > j$,

$$G_j^{n,2} \subset G_j^{n+1,2} \bigcup G_j^{n+1,1} \quad (4.19)$$

and for all $n > j+1$, $G_{j+1}^{n,2} \subset G_j^{n,2} \bigcup G_j^{n,1}$. This implies that

$$H_{j+1}^2 = \bigcup_{n>j+1} G_{j+1}^{n,2} \subset \bigcup_{n>j+1} (G_j^{n,2} \bigcup G_j^{n,1}) \subset \bigcup_{n>j} (G_j^{n,2} \bigcup G_j^{n,1}) = H_j^2 \bigcup H_j^1. \quad (4.20)$$

It follows that $H_n^2 \subset H_j^2 \bigcup H_j^1$ whenever $n > j$. As in Part (1) we have

$$1 = \mu(\limsup H_k^2) \stackrel{(4.20)}{\leq} \mu\left(\bigcap_{k=1}^{\infty} (H_k^2 \bigcup H_k^1)\right),$$

so that for all k , $\mu(H_k^2 \cap H_k^1) = 1$, and using Inclusion (4.19), this implies that $\lim_{n \rightarrow \infty} \mu(G_k^{n,2} \bigcup G_k^{n,1}) = 1$, so that $\lim_{n \rightarrow \infty} \mu(G_k^{n,2}) = 1$. Now one can construct a suitable sequence (n_k) as was done in (1).

Conversely, if for some (n_k) , $\sum_k \mu(G_{n_k}^{n_{k+1},2})$ diverges, then the converse of the Borel Cantelli lemma implies that almost all orders ω have at most 2 maximal paths. Since for each sequence (m_k) , $\sum_k \mu(G_{m_k}^{m_{k+1},1}) < \infty$, Part (1) tells us that $\mu(\mathcal{O}_B(1)) = 0$. The result follows. \square

If (F_n) , where $F_n = (f_{v,w}^{(n)})$, for $n \geq 1$, are the incidence matrices for B , consider the Markov matrices $M_n := (m_{v,w}^{(n)})$ where $m_{v,w}^{(n)} := \frac{f_{v,w}^{(n)}}{\sum_w f_{v,w}^{(n)}}$. Here $m_{v,w}^{(n)}$ represents the proportion of edges with range $v \in V_{n+1}$ that have source $w \in V_n$. Similarly, if (n_k) is a given sequence, consider for $j \geq 1$

$$F'_j := F_{n_{j+1}-1} \cdot F_{n_{j+1}-2} \cdot \dots \cdot F_{n_j+1} \quad (4.21)$$

and define the Markov matrices $M'_j = (m'_{v,w}{}^{(j)})$ as before. Proposition 4.2 tells us that the integer j such that $\mu(\mathcal{O}_B(j)) = 1$ depends only on the masses of the sets $G_{n_k}^{n_{k+1},j}$, as j and (n_k) vary. In turn, $\mu(G_{n_k}^{n_{k+1},j})$ depends only on the matrices M'_k where F'_k is defined as in (4.21), and is independent of the word ‘maximal’ which was used to define the sets $G_{n_k}^{n_{k+1},j}$. We have shown:

Corollary 4.3. *Let B be a finite rank d aperiodic Bratteli diagram. In the statement of Theorem 4.1, $j = j'$.*

The following corollary gives a sufficient condition for diagrams B where $\mu(\mathcal{O}_B(1)) = 1$. Note that this case includes all simple B with a bounded number of edges at each level. We use the notation of Equation 4.21.

Corollary 4.4. *Let B be a Bratteli diagram with incidence matrices (M_n) . Suppose there is some $\varepsilon > 0$, sequences (n_k) of levels and (w_k) of vertices (where $w_k \in V_{n_k}$), such that $m'_{v,w_k}^{(k)} \geq \varepsilon$ for all $k \in \mathbb{N}$ and $v \in V_{n_{k+1}}$. Then $\mu(\mathcal{O}_B(1)) = 1$.*

Proof. The satisfied condition implies that $\mu(G_{n_k}^{n_{k+1},1}) \geq \varepsilon^d$. Now apply Proposition 4.2. \square

Thus while in generally there is no algorithm, which, given a simple diagram B , finds the number of maximal paths that μ almost all orderings on B have; nevertheless Theorem 4.2 and Corollary 4.4 tell us that one can in principle find this number for a large class of diagrams.

Proposition 4.5 implies the following observation for simple diagrams. If B is a diagram for which $\mu(\mathcal{O}_B(j)) = 1$ with $j > 1$, then there is a meagreness of perfect orderings on B and hence dynamical systems defined on X_B . Part (2) of Proposition 4.5 implies an analogous statement for aperiodic diagrams.

Proposition 4.5. *Let B be a finite rank Bratteli diagram.*

1. *Suppose B is simple, of rank d . If $\mu(\mathcal{O}_B(1)) = 1$, then $\mu(\mathcal{P}_B) = 1$. If $\mu(\mathcal{O}_B(j)) = 1$ for some $j > 1$, then $\mu(\mathcal{P}_B) = 0$.*
2. *Suppose that B is aperiodic with q minimal components, and that its incidence matrices (F_n) have a strictly positive row R_n for each n , and where at least one entry in R_n tends to ∞ as $n \rightarrow \infty$. If $\mu(\mathcal{O}_B(q)) = 1$, then $\mu(\mathcal{P}_B) = 1$. If $\mu(\mathcal{O}_B(j)) = 1$ for some $j > q$, then $\mu(\mathcal{P}_B) = 0$.*

Proof. We remark that if $j = 1$, then clearly μ -almost all orderings are perfect. Suppose that B is simple and $\mu(\mathcal{O}_B(j)) = 1$ for some $j > 1$.

Fix $0 < \delta < 1/d$. Define, for $w \in V_{n-1}$,

$$P_n(w) := \{v \in V_n : m_{v,w}^{(n)} \geq \delta\};$$

then $V_n = \bigcup_{w: P_n(w) \neq \emptyset} P_n(w)$, and, if for infinitely many n , less than j of the $P_n(w)$'s are non-empty, then, for some $j' < j$, and some (n_k) , there is some ϵ such that $\mu(G_{n_k}^{n_{k+1},j'}) \geq \epsilon$ and Theorem 4.2 implies $\mu(\mathcal{O}_B(j')) = 1$ for $j' < j$, a contradiction. There is no harm in assuming that for fixed n , the sets $\{P_n(w) : P_n(w) \neq \emptyset\}$ are disjoint - if not we put $v \in P_n(w)$ where $m_{v,w}^{(n)}$ is maximal - and that there is some set $\{w_1, \dots, w_j\}$ of vertices such that $P_n(w_i) \neq \emptyset$ for each natural n and each $i = 1, \dots, j$. Pick $v_n^* \in V_n$ which has a large number of incoming edges. For ease of notation $v_n = v^*$. If all but finitely many vertices of the diagram are the range of a bounded number of edges, then Lemma 4.4 applies, implying that $\mu(\mathcal{O}_B(1)) = 1$, a contradiction. So we can assume that as n increases, v^* is the range of increasingly many edges.

Let \mathcal{E}_n be the event that

1. For each $v \in V_n$, the maximal and minimal edge with range v has source w_i whenever $v \in P_n(w_i)$;
2. For each n , there is a pair of consecutive edges with range $v^* \in V_n$, both having source w_i when $v^* \in P_n(w_i)$;
3. For each n , there is a pair of consecutive edges with range $v^* \in V_n$, the first having source w_i when $v^* \in P_n(w_i)$; the second having source $w_{i'}$ for some $i' \neq i$.

Then there is some δ^* such that $\mu(\mathcal{E}_n) \geq \delta^*$ for all large n . So for a set $\mathcal{O}_B(j)' \subset \mathcal{O}_B(j)$ of full measure, infinitely many of the events \mathcal{E}_n occur. For $\omega \in \mathcal{O}_B(j)'$, if $\omega \in \mathcal{E}_n$, then the extremal paths go through the vertices w_1, \dots, w_j at level n . Now an application of Part 2 of Proposition 3.2 implies that $\mathcal{O}_B(j)' \subset \mathcal{O}_B \setminus \mathcal{P}_B$.

To prove Part 2, first note that if B has q minimal components, then any ordering has at least q extremal pairs of paths. We assume that extremal paths come in pairs - otherwise the ordering is not perfect. If μ -almost all orderings have q maximal paths then necessarily each pair of extremal paths lives in a distinct minimal component of B , and μ almost all orderings belong to \mathcal{P}_B . Suppose that $\mu(\mathcal{O}_B(j)) = 1$ where $j > q$. Write

$$\mathcal{O}_B(j) = \bigcup_{\{(k_1, \dots, k_q) : \sum_{i=1}^q k_i \leq j\}} \mathcal{O}_B(j, \{(k_1, \dots, k_q)\})$$

where $\mathcal{O}_B(j, \{(k_1, \dots, k_q)\})$ is the set of orderings with k_i extremal pairs in the i -th minimal component. If for some i , $k_i > 1$, then by the argument in Part (1), $\mu(\mathcal{O}_B(j, \{(k_1, \dots, k_q)\})) = 0$. If $(k_1, \dots, k_q) = (1, \dots, 1)$ this means that there is at least one extremal pair of paths which lives outside the minimal components of B . Repeat the argument in Part 1, except that v^* must be chosen outside the union of the minimal components of B , and also such that at least one of the entries in $\{m_{v^*, v}^{(n)} : v \in V_n\}$ gets large as $n \rightarrow \infty$.

□

Example 4.6. It is not difficult to find a simple Bratteli diagram B where almost all orderings are not perfect, that is belong to $(\mathcal{P}_B)^c$. Let $V = \{v_1, v_2\}$, let $\sum_{n=1}^{\infty} m_{v_i, v_j}^{(n)} < \infty$ for $i \neq j$. Then for μ -almost all orderings, there is some K such that for $k > K$, the sources of the two maximal/minimal edges at level n are distinct - ie $\mu(\mathcal{O}_B(2)) = 1$. Note that here $\mu(\mathcal{O}_B(2)) = 1$ if and only if there are two probability measures on X_B which are invariant with respect to the tail equivalence relation. This is not in general true as the next example shows.

Example 4.7. This example appears in Section 4 of [FFT09]. Let

$$F_k := \begin{pmatrix} m_k & n_k & 1 \\ 0 & n_k - 1 & 1 \\ m_k - 1 & n_k & 1 \end{pmatrix}.$$

where the sequences (m_k) and (n_k) satisfy $3n_k + 1 \leq 2m_k \leq n_{k+1}$, which implies that they get large. The corresponding stochastic matrix satisfies

$$M_k \approx \begin{pmatrix} \frac{m_k}{m_k + n_k} & \frac{n_k}{m_k + n_k} & 0 \\ 0 & 1 & 0 \\ \frac{m_k}{m_k + n_k} & \frac{n_k}{m_k + n_k} & 0 \end{pmatrix},$$

and if we further require that $n_{k+1} \leq Cn_k$ for some $C \geq 4$, then $\frac{n_k}{m_k + n_k} \geq \frac{2}{2+C}$, so that by Corollary 4.4, $\mu(\mathcal{O}_B(1)) = 1$, while in [FFT09], it is shown that (a telescoping of) B has 2 probability measures which are invariant under the tail equivalence relation.

Example 4.8. Let

$$F_n := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

for n non-prime and

$$F_n := \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

if n is prime. Then if n is prime, given a vertex w , $m_{v,w}^{(n)} \geq 1/7$ either for $v = v_1$ or $v = v_5$. So $\mu(G_n^{n+1,2}) \geq (1/7)^7$. Also $\mu(G_n^{n+1,1}) = 0$ for each n . Theorem 4.2 implies that $j = 2$.

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